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The Control of Arbitrary Size Networks of Linear Systems via Graphon Limits: An Initial Investigation

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***Abstract*— To achieve control objectives for extremely complex and very large scale networks using standard methods is a challenging, if not intractable, task. In this paper, we propose a novel way to approximately control network systems which lie in a sequence with a well defined limit by the use of graphon theory and the theory of infinite dimensional systems. The general controllability problem is analyzed for the infinite system and then the control performance in terms of the upper bound for the L^2 state error between the limit system and the sequence of network systems is given. Finally, an example of the application of the minimum energy control methodology for network systems with sampled weightings is demonstrated.**

I. Introduction

Complex network systems are everywhere such as biological, gene, brain, citation, electric and social networks. The study of large scale networks has been the focus of much research. On one hand, scientists are studying the structural properties of networks, characterizing the structures and building models [1], [2] to mimic and reproduce certain structural properties of networks. On the other hand, researchers are studying networks of interacting dynamical systems to learn what collective behaviours would emerge over the interacting dynamical systems on a complex network. Over the past 15 years, topics such as network models, structures, controllability, observability, consensusability and synchronization in complex networks have been studied in system and network science [3], [4], [5], [6], [7], [8], [9], [10], [11].

To achieve control objectives for extremely complex and very large scale networks using standard methods is a challenging, if not intractable, task. In this work we propose a novel way to achieve approximate control for such networks by using the theory of graphons and infinite dimensional system theory. The proposed control strategy consists of the following steps: (1) Consider the general control problem of steering the states of each member of a sequence S of network systems $\{S^N; 1 \leq N \leq \infty\}$ to each of a sequence x_T of desired states $\{x_T^N; 1 \leq N \leq \infty\}$, where it is assumed that S converges to some limit system LS^∞ and x_T to some x_T^∞. (2) Specify the corresponding control problem CP^∞ for LS^∞ on $L^2[0,1]$ and choose a tolerance $\varepsilon > 0$. (3) Find the control law u^∞ solving CP^∞. (4) Then our main results in this paper together with the convergence of the x_T sequence yield N_ε such that $x_T^N(u^N)$ is within ε of x_T^∞ and of x_T^N for all $N \geq N_\varepsilon$.

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II. Preliminaries

A. Graphs, Adjacency Matrices and Pixel Pictures

The underlying structure of a network can be described by a graph $G = (V, E)$ specified by a vertex set V and an edge set E which represents the connections between vertices. An equivalent representation of a graph $G = (V, E)$ by a matrix called an *adjacency matrix* is defined to be the square $|V| \times |V|$ matrix A such that an element A_{ij} is one when there is an edge from vertex i to vertex j, and zero otherwise. If the graph is a weighted graph where edges are associated with weights, then the adjacency matrix has corresponding weighted elements.

Another representation of the adjacency matrix is given by a pixel diagram where the 0s are replaced by white squares and the 1s by black squares. The whole pixel diagram is presented in a unit square, so the square elements have sides of length $\frac{1}{n}$, where n is the number of vertices.

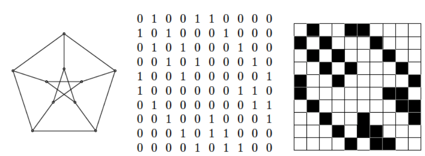

Fig. 1. Petersen Graph, Adjacency Matrix, Pixel Diagram [12]

B. Graphon

Graphon theory was introduced and developed in recent years by L. Lovász, B. Szegedy, C. Borgs, J. T. Chayes, V. T. Sós, and K. Vesztergombi among others in [13], [14], [15], [16], [12]. This work draws on graph theory (graphs are in fact special types of graphons), measure theory, probability, and functional analysis. A meaningful convergence with respect to the *cut metric* ([12]) is defined for sequences of dense and finite graphs. Graphons are then the limit objects of converging graph sequences. This concept is illustrated by a sequence of half graphs ([12]) represented by a sequence of pixel diagrams on the unit square converging to its limit in Fig. 2.

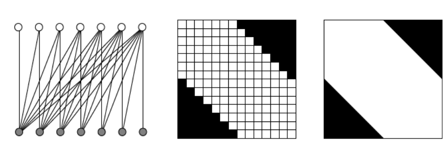

Fig. 2. Graph Sequence Converging to Its Limit [12]

The set of finite graphs endowed with the cut metric gives rise to a metric space, and the completion of this space is the *space of graphons*. Graphons are represented by bounded symmetric Lebesgue measurable functions $W : [0,1]^2 \rightarrow [0,1]$, which can be interpreted as weighted graphs on the vertex set $[0,1]$. We note that in some papers, for instance [17], the word "graphon" refers to symmetric, integrable functions from $[0,1]$ to R. In this paper, unless stated otherwise, the term "graphon" is used to refer to functions $\mathbf{W_1} : [0,1]^2 \rightarrow [-1,1]$ and $\tilde{\mathbf{G}}_\mathbf{1}^\mathbf{sp}$ denotes the space of graphons. Let $\tilde{\mathbf{G}}_\mathbf{0}^\mathbf{sp}$ represent the space of all graphons satisfying $\mathbf{W_0} : [0,1]^2 \rightarrow [0,1]$; let $\tilde{\mathbf{G}}^\mathbf{sp}$ denote the space of all symmetric measurable functions $\mathbf{W} : [0,1]^2 \rightarrow R$.

The cut norm of a graphon is then defined as

$$\|\mathbf{W}\|_\square = \sup_{M,T\subset[0,1]} |\int_{M\times T} \mathbf{W}(x,y)dxdy| \qquad (1)$$

with the supremum taking over all measurable subsets M and T of $[0,1]$. The inequalities between the different norms on a graphon $\mathbf{W}$ are

$$\|\mathbf{W}\|_\square \leq \|\mathbf{W}\|_1 \leq \|\mathbf{W}\|_2 \leq \|\mathbf{W}\|_\infty \leq 1. \qquad (2)$$

Denote the set of measure preserving bijections from $[0,1]$ to $[0,1]$ by $S_{[0,1]}$. The *cut metric* between two graphons $\mathbf{V}$ and $\mathbf{W}$ is then given by

$$d_\square(\mathbf{W},\mathbf{V}) = \inf_{\phi\in S_{[0,1]}} \|\mathbf{W}^\phi - \mathbf{V}\|_\square, \qquad (3)$$

where $\mathbf{W}^\phi(x,y) = \mathbf{W}(\phi(x),\phi(y))$. We see that the cut metric $d_\square(\cdot,\cdot)$ is given by measuring the maximum discrepancy between the integrals of two graphons over measurable subsets of $[0,1]$, then minimizing the maximum discrepancy over all possible measure preserving bijections.

Since the cut metric of two different graphons can be 0, strictly speaking it is not a metric. See [15], [18] for various characterizations of when the cut distance is 0. By identifying functions $\mathbf{V}$ and $\mathbf{W}$ for which $d_\square(\mathbf{V},\mathbf{W}) = 0$, we can construct the metric space $\mathbf{G}_\mathbf{1}^\mathbf{sp}$ which denotes the image of $\tilde{\mathbf{G}}_\mathbf{1}^\mathbf{sp}$ under this identification. Similarly we construct $\mathbf{G}_\mathbf{0}^\mathbf{sp}$ from $\tilde{\mathbf{G}}_\mathbf{0}^\mathbf{sp}$ and $\mathbf{G}^\mathbf{sp}$ from $\tilde{\mathbf{G}}^\mathbf{sp}$.

We define the L^2 *metric* for any graphons $\mathbf{W}$ and $\mathbf{V}$ as

$$d_{L^2}(\mathbf{W},\mathbf{V}) = \inf_{\phi\in S_{[0,1]}} \|\mathbf{W}^\phi - \mathbf{V}\|_2$$
$$= \inf_{\phi\in S_{[0,1]}} \left(\int_{[0,1]^2} |\mathbf{W}^\phi(x,y) - \mathbf{V}(x,y)|^2 dxdy\right)^{\frac{1}{2}}.$$

Then we can prove that for any two graphons $\mathbf{W}$ and $\mathbf{V}$

$$d_\square(\mathbf{W},\mathbf{V}) \leq d_{L^2}(\mathbf{W},\mathbf{V}). \qquad (4)$$

C. Compactness of the Graphon Space

Theorem 1: [12]. The space $(\mathbf{G}_\mathbf{0}^\mathbf{sp}, d_\square)$ is compact.

This remains valid if $\mathbf{G}_\mathbf{0}^\mathbf{sp}$ is replaced by any uniformly bounded subset of $\mathbf{G}^\mathbf{sp}$ closed in the cut metric [12].

Theorem 2: [12]. The space $(\mathbf{G}_\mathbf{1}^\mathbf{sp}, d_\square)$ is compact.

Sets in $\mathbf{G}_\mathbf{1}^\mathbf{sp}$ compact with respect to the L^2 metric are compact with respect to the cut metric. It follows immediately from (4) and Theorem 2 (or Theorem 1), if a graphon sequence is Cauchy in the L^2 metric then it is also a Cauchy sequence in the cut metric and under both metrics, the limits are identical in $\mathbf{G}_\mathbf{1}^\mathbf{sp}$ (or $\mathbf{G}_\mathbf{0}^\mathbf{sp}$).

D. Step Functions in the Graphon Space

Graphons generalize weighted graphs in the following sense (see [12]). A function $\mathbf{W} \in \mathbf{G}_\mathbf{1}^\mathbf{sp}$ is called a *step function* if there is a partition $Q = \{Q_1, ..., Q_k\}$ of $[0,1]$ into measurable sets such that $\mathbf{W}$ is constant on every product set $Q_i \times Q_j$. The sets Q_i are the *steps* of $\mathbf{W}$. For every weighted graph G (on node set $V(G)$), a step function $\mathbf{S_G} \in \mathbf{G}_\mathbf{1}^\mathbf{sp}$ is given as follows: partition $[0,1]$ into n measurable sets $Q_1, \cdots, Q_n$ of measure $\mu(Q_i) = \frac{\alpha_i}{\alpha_G}$, then for $x \in Q_i$ and $y \in Q_j$, we let $\mathbf{S_G}(x,y) = \beta_{ij}(G)$, where α_i denotes the node weight of i^{th} node, $\alpha(G) = \sum_i \alpha_i$ and $\beta_{ij}(G)$ denotes the weight of the edge from node i to node j. Evidently the function $\mathbf{S_G}$ depends on the labelling of the nodes of G. We define the *uniform partition* $P^N = \{P_1, P_2, ..., P_N\}$ of $[0,1]$ by setting $P_k = [\frac{k-1}{N}, \frac{k}{N}), k \in \{1, N-1\}$ and $P_N = [\frac{N-1}{N}, 1]$. Then $\mu(P_i) = \frac{1}{N}$, $i \in \{1,2,...,N\}$. Under the uniform partition, the step functions can be represented by the pixel diagram on the unit square.

E. Graphons as Operators

Following [12], a graphon $\mathbf{W} \in \mathbf{G}_\mathbf{1}^\mathbf{sp}$ can be interpreted as an operator $\mathbf{W} : L^2[0,1] \rightarrow L^2[0,1]$. The operation on $\mathbf{v} \in L^2[0,1]$ is defined as follows:

$$[\mathbf{W}\mathbf{v}](x) = \int_0^1 \mathbf{W}(x,\alpha)\mathbf{v}(\alpha)d\alpha, \qquad (5)$$

the operator product is then defined by

$$[\mathbf{U}\mathbf{W}](x,y) = \int_0^1 \mathbf{U}(x,z)\mathbf{W}(z,y)dz, \qquad (6)$$

where $\mathbf{U},\mathbf{W} \in \mathbf{G}_\mathbf{1}^\mathbf{sp}$. Note that if $\mathbf{U} \in \mathbf{G}_\mathbf{1}^\mathbf{sp}$ and $\mathbf{W} \in \mathbf{G}_\mathbf{1}^\mathbf{sp}$, then $\mathbf{U}\mathbf{W} \in \mathbf{G}_\mathbf{1}^\mathbf{sp}$ since for all $x,y \in [0,1]$

$$\begin{aligned} |[\mathbf{U}\mathbf{W}](x,y)| &= |\int_0^1 \mathbf{U}(x,z)\mathbf{W}(z,y)dz| \\ &\leq \int_0^1 |\mathbf{U}(x,z)\mathbf{W}(z,y)|dz \leq 1. \end{aligned} \qquad (7)$$

Consequently, the power $\mathbf{W}^n$ of an operator $\mathbf{W} \in \mathbf{G}_\mathbf{1}^\mathbf{sp}$ is defined as

$$\mathbf{W}^n(x,y) = \int_{[0,1]^n} \mathbf{W}(x,\alpha_1)\cdots\mathbf{W}(\alpha_{n-1},y)d\alpha_1\cdots d\alpha_{n-1},$$

with $\mathbf{W}^n \in \mathbf{G}_\mathbf{1}^\mathbf{sp}$ $(n \geq 1)$. $\mathbf{W^0}$ is formally defined as the identity operator on functions in $L^2[0,1]$, but we note that $\mathbf{W^0}$ is not a graphon.

F. The Graphon Unitary Operator Algebra

We have an operator algebra $\mathcal{G}_\mathcal{A}$ over the field R (see [19]) acting on elements of $L^2[0,1]$ as given by equation (5). By adjoining the identity element I to the algebra $\mathcal{G}_\mathcal{A}$ we obtain a unitary algebra $\mathcal{G}_{\mathcal{A}\mathcal{I}}$. The identity element I is defined as follows: for any $\mathbf{W} \in L^2[0,1]^2$

$$[\mathbf{I}\mathbf{W}](x,y) = \int_0^1 \mathbf{W}(z,y)\delta(x,z)dz = \mathbf{W}(x,y), \qquad (8)$$

where $\delta(\cdot,z)dz$ is the measure satisfying $\int_0^1 u(z)\delta(x,z)dz = u(x)$ for all $u \in L^2[0,1]$, and in particular $\int_0^1 \delta(x,z)dz = 1$.

The graphon unitary operator algebra $\mathcal{G}_{\mathcal{AI}}$ will be used in the definition of the controllability Gramian and the input operator. More specifically, we use the subset $\mathcal{G}^1_{\mathcal{AI}} = \{\mathcal{G}^1_{\mathcal{A}}, I\}$ where $\mathcal{G}^1_{\mathcal{A}}$ is the set in $\mathcal{G}_{\mathcal{A}}$ that corresponds to $\widetilde{\mathbf{G}}^{\mathbf{sp}}_{\mathbf{1}}$.

G. Graphon Differential Equations

Let X be a Banach space. A *linear operator* $A : D(A) \subset X \to X$ is *closed* if $\{(x, Ax) : x \in D(A)\}$ is closed in the product space $X \times X$(see [20]). $\mathcal{L}(X)$ denotes the Banach algebra of all linear continuous mappings $T : X \to X$. $L^p(a, b; X)$ denotes the Banach space of equivalent classes of strongly measurable (in the Böchner sense) mappings $[a, b] \to X$ that are p-integrable, $1 \le p < \infty$, with norm $\|f\|_{L^p(a,b;X)} = \left[\int_a^b |f(s)|^p ds\right]^{\frac{1}{p}}$. A mapping $S : R \to \mathcal{L}(X)$ is said to be a *strongly continuous semigroup* on X if the following properties hold:

(1) $S(0) = I, S(t + s) = S(t)S(s), \forall t, s \ge 0$
(2) for all $x \in X$, $S(\cdot)x$ is continuous on R.

A *uniformly continuous semigroup* is a strongly continuous semigroup S such that $\lim_{t\to 0^+} \|S(t) - I\| = 0$, with $\|\cdot\|$ as the operator norm on a Banach space. The *infinitesimal generator* A of a strongly continuous semigroup S is the linear operator in X defined by

$$Ax = \lim_{t\to 0^+} \frac{1}{t}[S(t)x - x], \quad \forall x \in \mathcal{D}(A), \tag{9}$$

where

$$\mathcal{D}(A) = \{x \in X : \text{ s.t. } \lim_{t\to 0^+} \frac{1}{t}[S(t)x - x] \text{ exists } \}.$$

Let $\mathbf{A} : [0, 1]^2 \to [-1, 1]$ be a graphon and hence a bounded and closed linear operator from $L^2[0, 1]$ to $L^2[0, 1]$. Following [21], $\mathbf{A}$ is the infinitesimal generator of the uniformly (hence strongly) continous semigroup $S_{\mathbf{A}}(t) := e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k \mathbf{A}^k}{k!}$. Therefore, the initial value problem of the graphon differential equation

$$\dot{\mathbf{y}}_{\mathbf{t}} = \mathbf{A}\mathbf{y}_{\mathbf{t}}, \quad \mathbf{y}_{\mathbf{0}} \in L^2[0, 1] \tag{10}$$

has a solution given by $\mathbf{y}_{\mathbf{t}} = e^{\mathbf{A}t}\mathbf{y}_{\mathbf{0}}$.

Theorem 3 ([19]): Let $\{\mathbf{A}_{\mathbf{N}}\}_{N=1}^{\infty}$ be a sequence of graphons such that $\mathbf{A}_{\mathbf{N}} \to \mathbf{A}_*$ as $N \to \infty$ in the L^2 metric. Then for all $\mathbf{x} \in L^2[0, 1]$, $e^{\mathbf{A}_{\mathbf{N}}t}\mathbf{x} \to e^{\mathbf{A}_* t}\mathbf{x}$ as $N \to \infty$ in the L^2 metric where the convergence is pointwise in time and uniform on any time interval $[0, T]$.

III. Network Systems and Their Limit Systems

A. Scaled Network Systems with Node Averaging Dynamics

Consider an interlinked network S^N of linear (symmetric) dynamical subsystems $\{S^N_i; 1 \le i \le N\}$, each with an n dimensional state space. Each subsystem S^N_i is uniquely associated to a vertex of the N node graph G_N whose undirected edges correspond to the dynamical interactions between the subsystems specified as below:

$$S^N_i : \quad \begin{aligned} &\dot{x}^i_t = \frac{1}{N}\sum_{j=1}^{N} \bar{A}_{N_{ij}} \frac{\Gamma}{n} x^j_t + \frac{1}{N}\sum_{j=1}^{N} \bar{B}_{N_{ij}} \frac{F}{n} u^j_t, \\ &x^i_t, u^i_t \in R^n, i \in \{1, ..., N\}, \end{aligned} \tag{11}$$

with $\bar{A}_N = [\bar{A}_{N_{ij}}], \bar{B}_N = [\bar{B}_{N_{ij}}] \in R^{N\times N}$ representing respectively the adjacency matrices of G_N and of the input graph. Assume $\Gamma, F \in R^{n\times n}$ are symmetric matrices. Then the (symmetric) linear dynamics for the network system $S^N(A_N, B_N, G_N)$ can be represented by

$$S^N : \quad \begin{aligned} &\dot{x}_t = A_N \circ x_t + B_N \circ u_t, \\ &x_t, u_t \in R^{nN}, \quad A_N, B_N \in R^{nN\times nN}, \end{aligned} \tag{12}$$

where $A_N = \bar{A}_N \otimes \Gamma$ denotes the symmetric (matrix weighted) adjacency matrix of G_N, $B_N = \bar{B}_N \otimes F$ denotes the linear input-to-state mapping, and $\circ$ denotes the so called averaging operator given by $A_N \circ x = \frac{1}{(nN)} A_N x$. Let $\mathcal{S} = \times_{N=1}^{\infty} \mathcal{S}^{\mathcal{N}}$ where $\mathcal{S}^{\mathcal{N}} = \cup_{A_N, B_N, G_N} S^N(A_N, B_N, G_N)$. For simplicity, we require the elements of A_N and B_N to be in $[-1, 1]$ for each N (note that in general A_N and B_N have elements that are bounded real numbers for which case we would achieve similar results). In addition, we note that if we take the supremum norm on vectors in R^{nN}, i.e. $\|x\|_\infty = \sup_i |x_i|$, and the corresponding $\circ$ operator norm of A, i.e. $\|A\|_{op} = \sup_{\|x\|_\infty \neq 0} \frac{\|A\circ x\|_\infty}{\|x\|_\infty}$, then $\|A\|_{op} \le 1$.

B. Network Systems with Node Averaging Dynamics Described by Step Functions in the Graphon Space

Let $\{(A_N; B_N)\}_{N=1}^{\infty} \in \mathcal{S}$ be a sequence of systems with the node averaging dynamics each of which is described according to (12). Let $|A_{Nij}| \le 1$ and $|B_{Nij}| \le 1$ for all $i, j \in \{1, ..., nN\}$. Let $\mathbf{A}^{[\mathbf{N}]}_{\mathbf{s}}, \mathbf{B}^{[\mathbf{N}]}_{\mathbf{s}} \in \mathbf{G}^{\mathbf{sp}}_{\mathbf{1}}$ be the step functions corresponding one-to-one to A_N and B_N; these are specified using the uniform partition P^{nN} of $[0, 1]$ by the following *matrix to step function mapping* M_G: for all $i, j \in \{1, 2, ..., nN\}$,

$$\mathbf{A}^{[\mathbf{N}]}_{\mathbf{s}}(x, y) := A_{Nij}, \quad \forall (x, y) \in P_i \times P_j, \tag{13}$$

and similar for $\mathbf{B}^{[\mathbf{N}]}_{\mathbf{s}}$.

Define a *piece-wise constant function* on R to be any function of the form $\sum_{k=1}^{l} \alpha_k \psi_{I_k}$ where $\alpha_1, ..., \alpha_l$ are complex numbers and each I_k is a bounded interval (open, closed, or half-open). Let $L^2_{pwc}[0, 1]$ denote the space of piece-wise constant $L^2[0, 1]$ functions under the uniform partition P^{nN}.

Let $\mathbf{u}^{\mathbf{s}}_{\mathbf{t}} \in L^2_{pwc}[0, 1]$ correspond one-to-one to $u_t \in R^{nN}$ via the following *vector to step function mapping* also denoted by M_G: for all $i \in \{1, ..., nN\}$,

$$\mathbf{u}^{\mathbf{s}}_{\mathbf{t}}(\alpha) := u_t(i), \quad \forall \alpha \in P_i, \tag{14}$$

and $\mathbf{x}^{\mathbf{s}}_{\mathbf{t}} \in L^2_{pwc}[0, 1]$ similarly correspond one-to-one to $x_t \in R^{nN}$.

Lemma 1 ([19]): The trajectories of the system in (12) correspond one-to-one under the mapping M_G to the trajectories of the system

$$\begin{aligned} &\dot{\mathbf{x}}^{\mathbf{s}}_{\mathbf{t}} = \mathbf{A}^{[\mathbf{N}]}_{\mathbf{s}} \mathbf{x}^{\mathbf{s}}_{\mathbf{t}} + \mathbf{B}^{[\mathbf{N}]}_{\mathbf{s}} \mathbf{u}^{\mathbf{s}}_{\mathbf{t}}, \\ &\mathbf{x}^{\mathbf{s}}_{\mathbf{t}}, \mathbf{u}^{\mathbf{s}}_{\mathbf{t}} \in L^2_{pwc}[0, 1], \mathbf{A}^{[\mathbf{N}]}_{\mathbf{s}}, \mathbf{B}^{[\mathbf{N}]}_{\mathbf{s}} \in \mathbf{G}^{\mathbf{sp}}_{\mathbf{1}} \subset \mathcal{G}^1_{\mathcal{AI}} \end{aligned} \tag{15}$$

with graphon operations defined according to (5).

C. Limits of Sequences of Network Systems

Now the sequence of network systems with the node averaging dynamics can be described by the sequence of step function operators as $\{(\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}; \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]})\}_{N=1}^{\infty}$. Let the graphon sequences $\{\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}\}$ and $\{\mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]}\}$ be Cauchy sequences of step functions in $L^2[0,1]^2$. Due to the completeness of $L^2[0,1]^2$, the respective graphon limits $\mathbf{A}$ and $\mathbf{B}$ exist and these will then necessarily be the limits in the cut metric (see [12]).

IV. The Limit Graphon System and Its Properties

A. Infinite Dimensional Graphon Systems

We follow [20] and specialize the Hilbert space of states H and the Hilbert space of controls U appearing there to the space $L^2(R; L^2[0,1])$. We formulate an infinite dimensional linear system as follows:

$$LS^{\infty}: \quad \dot{\mathbf{x}}_{\mathbf{t}} = \mathbf{A}\mathbf{x}_{\mathbf{t}} + \mathbf{B}\mathbf{u}_{\mathbf{t}}, \quad \mathbf{x}_{\mathbf{0}} \in L^2[0,1], \tag{16}$$

where $\mathbf{A}, \mathbf{B} \in \mathbf{G}_{\mathbf{1}}^{\mathbf{sp}} \subset \mathcal{G}_{\mathcal{AI}}^{\mathbf{1}}$ are graphons, and hence bounded operators on $L^2[0,1]$, $\mathbf{x}_{\mathbf{t}} \in L^2[0,1]$ is the system state at time t and $\mathbf{u}_{\mathbf{t}} \in L^2[0,1]$ is the control input at time t.

B. Uniqueness of the Solution

A solution $\mathbf{x}_{(\cdot)} \in L^2(R; L^2[0,1])$ is a *(mild) solution* of (16) if $\mathbf{x}_{\mathbf{t}} = e^{(t-a)\mathbf{A}}\mathbf{x}_{\mathbf{a}} + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}\mathbf{u}_{\mathbf{s}} ds$ for all a and t in R such that $a \le t$ (see [20]). Following [20] the assumptions on the operators $\mathbf{A}$ and $\mathbf{B}$ are

$$(H1) \quad \begin{cases} \text{(i)} & \mathbf{A} \text{ generates a strongly continuous} \\ & \text{semigroup } e^{t\mathbf{A}} \text{ on } L^2[0,1], \\ \text{(ii)} & \mathbf{B} \in \mathcal{L}(L^2[0,1]; L^2[0,1]), \end{cases}$$

where the Hilbert space U(control space) in the present case is $L^2[0,1]$. Under assumption $(H1)$, the system (16) has a unique solution $\mathbf{x} \in C([0,T]; L^2[0,1])$ for any $\mathbf{x_0} \in L^2[0,1]$ and any $\mathbf{u} \in L^2(0,T; L^2[0,1])$.

Theorem 4: The graphon system LS^{∞} in Eq. (16) has a unique solution $\mathbf{x} \in C([0,T]; L^2[0,1])$ for any $\mathbf{x_0} \in L^2[0,1]$ and any $\mathbf{u} \in L^2(0,T; L^2[0,1])$.

Proof: Since A as a graphon operator generates a uniformly continuous semigroup, $H1(i)$ is satisfied. Moreover B as a graphon operator is bounded and hence a continous linear mapping from control space $U = L^2[0,1]$ to the state space $L^2[0,1]$ satisfying $H1(ii)$. Therefore the system (16) has a unique solution $\mathbf{x} \in C([0,T]; L^2[0,1])$ for any $\mathbf{x_0} \in L^2[0,1]$ and any $\mathbf{u} \in L^2(0,T;U)$. □

C. Controllability

The system $(\mathbf{A}; \mathbf{B})$ is *controllable* on $[0,T]$ if for any initial state $\mathbf{x}_0 \in L^2[0,1]$ and any target state $\mathbf{x_T} \in L^2[0,1]$, there exists a control $\mathbf{u} \in L^2(0,T;U)$ driving the system from $\mathbf{x_0}$ to $\mathbf{x_T}$, i.e. $\mathbf{x_T} = e^{\mathbf{A}T}x_0 + \int_0^T e^{\mathbf{A}(T-t)}\mathbf{B}\mathbf{u_t} dt$.

A necessary and sufficient condition for the system $(\mathbf{A}; \mathbf{B})$ to be controllable (called exact controllability in this case in [22]) on $[0,T]$ is

$$(\mathbf{W_T}h, h) \ge c_T \|h\|^2, \quad \forall h \in L^2[0,1],$$

where $\mathbf{W_T} = \int_0^T e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t} dt$ is the *controllability Gramian operator* (see [20], [22]), $c_T > 0$ and $\|\cdot\|$ is the $L^2[0,1]$ norm. $\mathbf{W_T}$ as an operator in the graphon unitary operator algebra acts on any $L^2[0,1]$ function h as follows:

$$\forall \alpha \in [0,1], \quad [\mathbf{W_T}h](\alpha) = \int_0^1 \mathbf{W_T}(\alpha, z)h(z)dz. \tag{17}$$

We note that in the present case, a state $\mathbf{x}$ in the state space is an equivalence class of $L^2[0,1]$ functions which are zero distance from any representative of the class.

V. Limit Graphon Control of Large-scale Networks

A. Approximation of $L^2[0,1]$ Input Functions via Piece-wise Constant Functions

Theorem 5: [23](p.198) Let λ be any measure on R and let $1 \le p < \infty$. Then piece-wise constant functions on R form a dense subset of $L^p(R, B_\lambda, \lambda)$.

Therefore piece-wise constant functions can approximate L^2 functions arbitrarily well. In our case, we want to approximate the control input $\mathbf{u_t}(\cdot) \in L^2[0,1], 0 \le t \le T$, through a piece-wise constant function in $L^2[0,1]$ denoted by $\mathbf{u_t^N}(\cdot)$. Specifically, the approximation of input $\mathbf{u_t}(\cdot)$ by $\mathbf{u_t^N}(\cdot)$ with the partition $Q = \{Q_1, Q_2, \cdots, Q_{nN}\}$ of $[0,1]$ is given as follows: for all $Q_i, i \in \{1, 2, \ldots, nN\}$,

$$\mathbf{u_t^N}(\alpha) = \frac{1}{\mu(Q_i)} \int_{Q_i} \mathbf{u_t}(\beta) d\beta, \quad \forall \alpha \in Q_i, \tag{18}$$

where $\mu(Q_i)$ denotes the measure of Q_i.

B. Limit Control for Network Systems with General Graphon Input Mappings

Consider a finite dimensional system $(A_N; B_N)$ with node averaging dynamics as in (12) and $(\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}; \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]})$ as its equivalent step function system according to (13).

Theorem 6 ([19]): Consider a sequence of network systems $\{(\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}; \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]})\}$ converging to a graphon system $(\mathbf{A}; \mathbf{B})$, i.e. $\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]} \to \mathbf{A}$ and $\mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]} \to \mathbf{B}$ in the L^2 metric as $N \to \infty$. Assume that $(\mathbf{A}; \mathbf{B})$ is controllable and that for some N_0, (A^N, B^N) is controllable for all $N \ge N_0$. Then for any $T > 0$, $N \ge N_0$ and for any $\mathbf{x}$ in $L^2[0,1]$:
(1) there exists $\mathbf{v} \in L^2[0,1]$ such that $\mathbf{x_T}(\mathbf{v}) = \mathbf{x}$, and for each $N \ge N_0$ there exists a control $\mathbf{v}^{[\mathbf{N}]}$ for $(\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}; \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]})$ approximating the control $\mathbf{v}$ for $(\mathbf{A}; \mathbf{B})$ such that

$$\begin{aligned} &\|\mathbf{x_T}(\mathbf{v}) - \mathbf{x_T^N}(\mathbf{v}^{[\mathbf{N}]})\|_2 \\ &\quad \le \|\mathbf{A}_\Delta^\mathbf{N}\|_2 \|\mathbf{B}\|_2 \int_0^T e^{T-\tau}(T-\tau) \cdot \|\mathbf{v}_\tau\|_2 d\tau \\ &\quad + \|\mathbf{B}_\Delta^\mathbf{N}\|_2 \int_0^T e^{(T-\tau)\|\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}\|_2} \cdot \|\mathbf{v}_\tau\|_2 d\tau, \end{aligned} \tag{19}$$

where $\mathbf{A}_\Delta^\mathbf{N} = \mathbf{A} - \mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}$ and $\mathbf{B}_\Delta^N = \mathbf{B} - \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]}$;
(2) furthermore, $\lim_{N \to \infty} \|\mathbf{x_T}(\mathbf{v}) - \mathbf{x_T^N}(\mathbf{v}^{[\mathbf{N}]})\|_2 = 0$.

The control approximation is given in the following: $\mathbf{v}_{\mathbf{t}}^{[\mathbf{N}]}(\alpha) = nN \int_{P_i} \mathbf{v_t}(\beta) d\beta$, for all $\alpha \in P_i$, with the uniform partition $P^{nN} = \{P_1, \cdots, P_{nN}\}$. Then according to the M_G mapping, the control law $v^N(\cdot)$ for the finite network system $(A_N; B_N)$ is given by

$$v_t^N(i) = \mathbf{v}_{\mathbf{t}}^{[\mathbf{N}]}(\alpha), \quad \forall i \in \{1, ..., nN\}, \forall \alpha \in P_i, t \in [0,T].$$

C. Limit Control for Network Systems with the Identity Input Mapping

In fact, the control input mapping $\mathbf{B}$ is not limited to be a graphon mapping. As long as the control input map is a continous mapping from $L^2[0,1]$ to $L^2[0,1]$, the existence and uniqueness of a solution are guaranteed. The identity operator $\mathbf{I}$ is a continous mapping from $L^2[0,1]$ to $L^2[0,1]$ and hence the system $(\mathbf{A};\mathbf{I})$ has a unique solution. We note that while the identity operator $\mathbf{I}$ may be represented by a positive measure on the diagonal in $[0,1]^2$ and it may be treated as an element of $L^1[0,1]^2$, it is not an element of $L^2[0,1]^2$ and hence not in $\mathbf{G_1^{sp}}$.

Consider a finite dimensional system $(A_N;I_N)$ with node averaging dynamics and $(\mathbf{A_s^{[N]}};\mathbf{I})$ as its equivalent step function system according to (13).

Lemma 2 ([19]): Suppose $\mathbf{A_s^{[N]}} \to \mathbf{A}$ in $L^2[0,1]^2$ metric as $N\to\infty$. Then there exists a control $\mathbf{u^{[N]}}$ for $(\mathbf{A_s^{[N]}};\mathbf{I})$ approximating the control $\mathbf{u}$ for $(\mathbf{A};\mathbf{I})$ such that

$$\begin{aligned}\|\mathbf{x_T}(\mathbf{u})-\mathbf{x_T^N}(\mathbf{u^{[N]}})\|_2 \leq &\|\mathbf{A_\Delta^N}\|_2\int_0^T e^{T-\tau}(T-\tau)\|\mathbf{u}_\tau\|_2 d\tau\\ &+\|\int_0^T[\mathbf{u}_\tau-\mathbf{u}_\tau^{[\mathbf{N}]}]d\tau\|_2, \end{aligned}\qquad (20)$$

where $\mathbf{A_\Delta^N}=\mathbf{A}-\mathbf{A_s^{[N]}}$. The control approximation is given by $\mathbf{u_t^{[N]}}(\alpha)=nN\int_{P_i}\mathbf{u_t}(\beta)d\beta$, for all $\alpha\in P_i$, with the uniform partition $P^{nN}=\{P_1,\cdots,P_{nN}\}$.

Here the control law $u^N(\cdot)$ for the finite network system $(A_N;I_N)$ is given by

$$u_t^N(i)=\mathbf{u_t^{[N]}}(\alpha),\quad \forall i\in\{1,...,nN\},\forall\alpha\in P_i, t\in[0,T].$$

Note that u^N always exists by definition since the control approximation given by (18) uses the same uniform partition as the step function approximation in the graphon space.

D. Limit Graphon Control Strategy (LGCS)

The proposed control strategy consists of four steps:

(1) Consider the control problem of steering the states of each member of $\{(A_N;B_N)\}_{N=1}^\infty\in\mathcal{S}$ to each of a sequence of desired states $\{x_T^N\in R^{nN}\}_{N=1}^\infty$. Let $\{(\mathbf{A_s^{[N]}};\mathbf{B_s^{[N]}})\in\mathbf{G_1^{sp}}\times\mathbf{G_1^{sp}}\}_{N=1}^\infty$ be the sequence of step function systems equivalent to $\{(A_N;B_N)\}_{N=1}^\infty\in\mathcal{S}$ under the mapping M_G and assume that it converges to the graphon system $(\mathbf{A};\mathbf{B})\in\mathbf{G_1^{sp}}\times\mathbf{G_1^{sp}}$ in the L^2 metric. Let $\{\mathbf{x_T^N}\in L^2[0,1]\}_{N=1}^\infty$ be the image of $\{x_T^N\in R^{nN}\}_{N=1}^\infty$ under M_G, which is assumed to converge to some $\mathbf{x_T^\infty}\in L^2[0,1]$ in the $L^2[0,1]$ norm.

(2) Specify the corresponding state to state control problem CP^∞ for $(\mathbf{A};\mathbf{B})\in\mathbf{G_1^{sp}}\times\mathbf{G_1^{sp}}$ on $L^2[0,1]$ and choose a tolerance $\varepsilon>0$.

(3) Find a control law $\mathbf{u}^\infty:=\{\mathbf{u}_\tau\in L^2[0,1],\tau\in[0,T]\}$ solving CP^∞.

(4) Then the convergence of the sequence $\{\mathbf{x_T^N}\in L^2[0,1]\}_{N=1}^\infty$, and Theorem 6 yield N_ε such that $\mathbf{x_T^N}(\mathbf{u^N})$ is within ε of $\mathbf{x_T^\infty}$ and of $\mathbf{x_T^N}$ for all $N\geq N_\varepsilon$ under the $L^2[0,1]$ norm.

VI. Minimum Energy Limit Graphon Control

A specific control law used in Step (2) of the LGCS is described in this section.

A. Minimum Energy Control of Infinite Dimensional Systems

Define the energy cost by the control over the time horizon $[0,t]$ as $J(\mathbf{u})=\int_0^t\|\mathbf{u}_\tau\|^2d\tau,(t>0)$. The objective is to drive the system from some initial state $\mathbf{x_0}\in L^2[0,1]$ to some target state $\mathbf{x_T}\in L^2[0,1]$ using minimum control energy. A function $\mathbf{u}^*\in L^2(0,t;U)$ is called an optimal control if $J(\mathbf{u}^*)\leq J(\mathbf{u})$, for all $\mathbf{u}\in L^2(0,t;U)$ that drive the system from $\mathbf{x_0}$ to $\mathbf{x_T}$.

Theorem 7 ([19]): If the graphon system $(\mathbf{A};\mathbf{B})$ with $\mathbf{W_T}$ as its graphon controllability Gramian operater is exactly controllable, then the inverse operator $\mathbf{W_T^{-1}}$ exists and is a bounded operator.

B. Minimum Energy Control Law

Assume the system $(\mathbf{A};\mathbf{B})$ is exactly controllable, then $\mathbf{W_T^{-1}}$ exists and the optimal control law that achieves the minimum energy control is given by

$$\mathbf{u}_\tau^*=\mathbf{B}^Te^{\mathbf{A}^T(T-\tau)}\mathbf{W_T^{-1}}(\mathbf{x_T}-e^{\mathbf{A}(T)}\mathbf{x_0}),\quad \tau\in[0,T]. \qquad (21)$$

The minimum energy for controlling the system in time horizon $[0,T]$ is

$$\|\mathbf{u}\|_2^2=[\mathbf{x_T}-e^{\mathbf{A}(T)}\mathbf{x_0}]^T\mathbf{W_T^{-1}}[\mathbf{x_T}-e^{\mathbf{A}(T)}\mathbf{x_0}]. \qquad (22)$$

VII. Control of Network Systems with Sampled Weightings

Consider a network system evolving according to the node averaging dynamics with G_N describing the dynamic interactions. Suppose each node has an independent input channel. Denote the system by $(A_N;I_N)$, where A_N is the adjacency matrix of G_N and I_N is the identity input mapping. The network system $(A_N;I_N)$ with node averaging dynamics is therefore described by

$$\dot{x}_t^i=\frac{1}{N}\sum_{j=1}^N A_{Nij}x_t^j+u_t^i,\quad x_t^i,u_t^i\in R, i\in\{1,...,N\}, \qquad (23)$$

where A_{Nij} is sampled from the graphon $\mathbf{U}$ as follows: (1) Uniformly sample N distinct points from $[0,1]$. Sort the sample points in the decreasing order of their values and label them from node 1 to node N. Denote the node set by V_N and the value of node $i\in V_N$ by v_i. (2) Connect the nodes $i,j\in V_N$ with edge weight $\mathbf{U}(v_i,v_j)$ to generate the network G_N. Then $A_{Nij}=\mathbf{U}(v_i,v_j)$ is the ij^{th} element of the adjacency matrix of G_N.

If $\mathbf{U}$ is almost everywhere continuous, then the step function $\mathbf{A_s^{[N]}}$ of $A_N=[A_{Nij}]$ converges to $\mathbf{U}$ in the $L^1[0,1]^2$ metric as $N\to\infty$, that is

$$d_{L^1}(\mathbf{A_s^{[N]}},U):=\inf_{\varphi\in S_{[0,1]}}\|\mathbf{A_s^{[N]}}-U^\varphi\|_1\to 0 \qquad (24)$$

as $N\to\infty$, where $S_{[0,1]}$ denotes the set of measure preserving bijections from $[0,1]$ to $[0,1]$ as before. Further suppose $\mathbf{U}$ is bounded, then (24) implies $d_{L^2}(\mathbf{A_s^{[N]}},\mathbf{U})\to$

0, as $N \to \infty$. It follows that if $\mathbf{U}$ is almost everywhere continuous and bounded, then we can apply LGCS to control the sampled network systems.

As an example, we consider the case where $\mathbf{U}(x, y) = 1 - \max(x, y)$ for all $x, y \in [0, 1]$ and solve the minimum energy control problem of driving the states of the network system $(A_N; I_N)$ to a Gaussian terminal state distribution x_T^N from the origin over the time horizon $[0, T]$.

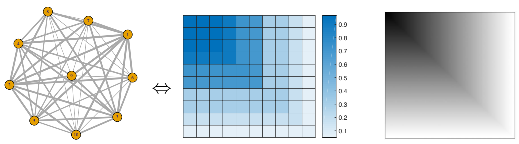

Fig. 3. Weighted Graph Generated from $\mathbf{U}$, its Stepfunction and Limit Graphon $\mathbf{U}$

The systems $(\mathbf{U}; \mathbf{I})$ is controllable and the forward controllability Gramian operator is given by

$$\mathbf{W}_T = \int_0^T e^{\mathbf{U}(T-s)} e^{\mathbf{U}^T(T-s)} ds = \int_0^T e^{2\mathbf{U}(T-s)} ds. \quad (25)$$

The minimum energy control for $(\mathbf{U}; \mathbf{I})$ is given by

$$\mathbf{u}_\tau^* = e^{\mathbf{U}^T(T-\tau)} \mathbf{W}_\mathbf{T}^{-1} \mathbf{x}_\mathbf{T}, \quad \tau \in [0, T], \quad (26)$$

Then the control law $u_{(\cdot)}^N$ for a network system $(A_N; I_N)$ generated by $\mathbf{U}$ comes from the following approximation: $u_\tau^N(i) = N \int_{P_i} \mathbf{u}_\tau^*(\beta) d\beta, \tau \in [0, T]$, where P_i is the i^{th} element of the uniform partition P^N of $[0, 1]$. The error $\|\mathbf{x}_\mathbf{T}(\mathbf{u}) - \mathbf{x}_\mathbf{T}^\mathbf{N}(u^{[\mathbf{N}]})\|_2$ is bounded as in (20) and converges to 0 as $N \to \infty$. The result of a simulation with a network system with 50 nodes using the proposed approximate control is shown as below.

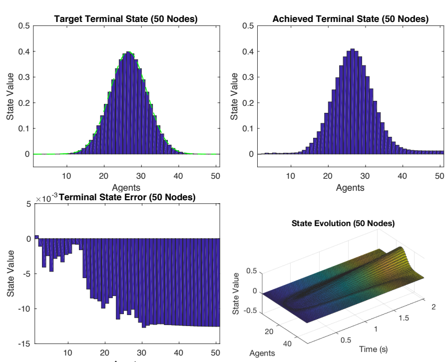

Fig. 4. Target State, Achieved State, Terminal State Error and State Evolution Over Time

VIII. Conclusion

We propose a method to approximately control network systems with node averaging dynamics using the inherent structural limit described by graphons. Important aspects requiring further investigations include: (1) the application of the proposed limit graphon control strategy to asymmetric network systems where the interactions of dynamics are described by directed networks; (2) other control problems such as LQR problem.

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