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Graphon Linear Quadratic Regulation of Large-scale Networks of Linear Systems

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Abstract— **We propose a graphon regulation methodology to solve linear quadratic regulator (LQR) problems for complex networks of dynamical systems following the formulation initiated in [1]. Conditions for the exact and approximate controllability of graphon dynamical systems are investigated. Approximation schemes are then developed to obtain finite dimensional LQR control laws which are utilized on large-scale network systems and for which the convergence properties are established. Finally, an example of the application of graphon-LQR control to networks of dynamical systems is given in which the Riccati equation of the limit graphon system is solved explicitly.**

I. Introduction

The problem of controlling complex networks of dynamical systems emerges in many applications ranging from the Internet of Things (IoT), smart grids, neuronal networks, food webs, social networks, to stock market networks, and it has received a great deal of attention during the past two decades. Related research studies have been focusing on (a) analysis problems such as controllability [2], observability [3] and control energy metric [4], etc., and (b) synthesis problems with simple objectives or simple control laws such as consensus [5], synchronization [6], [7], flocking [8], ensemble control [9], etc. In spite of these contributions, the understanding of complex networks of dynamical systems and furthermore the creation of a control theory for such systems still pose fundamental problems. One of the reasons that controlling these networks is so difficult or even intractable lies in their large (or even continuously growing) sizes and the complexity of their interconnections. To address these issues, we proposed the graphon control methodology [1] in which graphon theory [10]–[12] and the theory of infinite dimensional systems [13], [14] are combined to model complex networks of dynamical systems and to design control laws.

In the work of [1], the minimum energy state-to-state control problem for networks of linear dynamical systems is formulated and solved. In this paper, we further develop the graphon control theoretic method to solve regulation problems on complex network systems with linear quadratic costs. In addition, we investigate conditions for the exact and approximate controllability of graphon dynamical systems.

Consider the problem of applying linear quadratic regulation to each member of a sequence of networks. The proposed graphon regulation strategy consists of the following steps: (1) Identify a graphon either as a limit of the sequence of networks as the number of nodes grows without bound, or as an approximation to the sequence in the finite sequence case. (2) Solve the corresponding LQR problem for the graphon system by solving the graphon system Riccati equation. (3) Approximate the Riccati equation solution for the graphon system so as to generate approximated control laws for the original sequence of finite network systems.

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II. Preliminaries

A. Graphon

Graphons can be considered as the limit objects of convergent graph sequences under the so-called *cut metric* [12]. This concept is illustrated by a sequence of half graphs ([12]) represented by a sequence of pixel diagrams on the unit square converging to its limit in Fig. 1. In this paper,

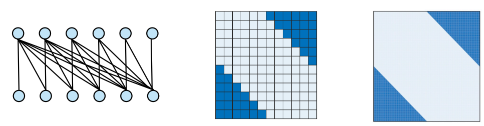


Fig. 1. Graph Sequence Converging to Its Limit

unless stated otherwise, the term "graphon" is used to refer to symmetric Lebesgue measurable functions $\mathbf{W_1} : [0,1]^2 \to [-1,1]$ and $\tilde{\mathbf{G}}_\mathbf{1}^\mathbf{sp}$ denotes the space of graphons. Let $\tilde{\mathbf{G}}_\mathbf{0}^\mathbf{sp}$ represent the space of all graphons satisfying $\mathbf{W_0} : [0,1]^2 \to [0,1]$; let $\tilde{\mathbf{G}}^\mathbf{sp}$ denote the space of all symmetric measurable functions $\mathbf{W} : [0,1]^2 \to \mathcal{R}$. The cut norm of a graphon is then defined as

$$\|\mathbf{W}\|_\Box = \sup_{M,T\subset[0,1]} \left|\int_{M\times T} \mathbf{W}(x,y)dxdy\right| \tag{1}$$

with the supremum taking over all measurable subsets M and T of $[0,1]$. Evidently, the following inequalities hold between norms on a graphon $\mathbf{W}$

$$\|\mathbf{W}\|_\Box \le \|\mathbf{W}\|_1 \le \|\mathbf{W}\|_2 \le \|\mathbf{W}\|_\infty \le 1. \tag{2}$$

Denote the set of measure preserving bijections from $[0,1]$ to $[0,1]$ by $S_{[0,1]}$. The *cut metric* between two graphons $\mathbf{V}$ and $\mathbf{W}$ is then given by

$$\delta_\Box(\mathbf{W},\mathbf{V}) = \inf_{\phi\in S_{[0,1]}} \|\mathbf{W}^\phi - \mathbf{V}\|_\Box, \tag{3}$$

where $\mathbf{W}^\phi(x,y) = \mathbf{W}(\phi(x),\phi(y))$. By identifying functions $\mathbf{V}$ and $\mathbf{W}$ for which $\delta_\Box(\mathbf{V},\mathbf{W}) = 0$, we can construct the space $\mathbf{G}_\mathbf{1}^\mathbf{sp}$ which denotes the image of $\tilde{\mathbf{G}}_\mathbf{1}^\mathbf{sp}$ under this identification. Similarly we construct $\mathbf{G_0^{sp}}$ from $\tilde{\mathbf{G}}_\mathbf{0}^\mathbf{sp}$ and $\mathbf{G^{sp}}$ from $\tilde{\mathbf{G}}^\mathbf{sp}$.

We define the L^2 *metric* distance between any graphons $\mathbf{W}$ and $\mathbf{V}$ as

$$d_{L^2}(\mathbf{W},\mathbf{V}) = \|\mathbf{W}-\mathbf{V}\|_2 \tag{4}$$

and the δ_2 *metric* as $\delta_2(\mathbf{W},\mathbf{V}) = \inf_{\phi\in S_{[0,1]}} d_{L^2}(\mathbf{W}^\phi,\mathbf{V})$. Similarly, we define the L^1 *metric* as

$$d_{L^1}(\mathbf{W},\mathbf{V}) = \|\mathbf{W}-\mathbf{V}\|_1 \tag{5}$$

and the δ_1 *metric* as $\delta_1(\mathbf{W},\mathbf{V}) = \inf_{\phi\in S_{[0,1]}} d_{L^1}(\mathbf{W}^\phi,\mathbf{V})$.

For any two graphons $\mathbf{W}$ and $\mathbf{V}$ the following inequalities hold immediately:

$$\delta_\Box(\mathbf{W},\mathbf{V}) \le \delta_1(\mathbf{W},\mathbf{V}) \le \delta_2(\mathbf{W},\mathbf{V}) \le d_{L^2}(\mathbf{W},\mathbf{V}). \tag{6}$$

The δ_2 (or δ_1) metric and $\delta_\Box$ metric share the same equivalence classes [12, Corollary 8.14]. Clearly, the δ_2 (or δ_1) metric is also well defined on $\mathbf{G_1^{sp}}$.

B. Compactness of the Graphon Space

Theorem 1 ([12]). The space $(\mathbf{G_0^{sp}}, \delta_\Box)$ is compact.

This remains valid if $\mathbf{G_0^{sp}}$ is replaced by any uniformly bounded subset of $\mathbf{G^{sp}}$ closed in the cut metric [12].

Theorem 2 ([12]). The space $(\mathbf{G_1^{sp}}, \delta_\Box)$ is compact.

Sets in $\mathbf{G_1^{sp}}$ (or $\mathbf{G_0^{sp}}$) compact with respect to the δ_2 metric are compact with respect to the cut metric. It follows immediately from (6) and Theorem 2 (or Theorem 1), if a graphon sequence is Cauchy in the δ_2 metric then it is also a Cauchy sequence in the cut metric and under both metrics, the limits are identical in $\mathbf{G_1^{sp}}$ (or $\mathbf{G_0^{sp}}$).

C. Step Functions in the Graphon Space

Graphons generalize weighted graphs in the following sense (see also [12]). A function $\mathbf{W} \in \mathbf{G_1^{sp}}$ is a *step function* if there is a partition $Q = \{Q_1, ..., Q_k\}$ of $[0,1]$ into measurable sets such that $\mathbf{W}$ is constant on every product set $Q_i \times Q_j$. The sets Q_i are the *steps* of $\mathbf{W}$. For every weighted graph G (on node set $V(G)$), a step function $\mathbf{S_G} \in \mathbf{G_1^{sp}}$ is given as follows: partition $[0,1]$ into n measurable sets $Q_1, \cdots, Q_n$ of measure $\mu(Q_i) = \frac{\alpha_i}{\alpha_G}$, then for $x \in Q_i$ and $y \in Q_j$, we let $\mathbf{S_G}(x,y) = \beta_{ij}(G)$, where α_i denotes the node weight of i^{th} node, $\alpha(G) = \sum_i \alpha_i$ and $\beta_{ij}(G)$ denotes the weight of the edge from node i to node j (i.e., β_{ij} is the ij^{th} entry in the adjacency matrix of G). Evidently the function $\mathbf{S_G}$ depends on the labelling of the nodes of G. We define the *uniform partition* $P^N = \{P_1, P_2, ..., P_N\}$ of $[0,1]$ by setting $P_k = [\frac{k-1}{N}, \frac{k}{N}), k \in \{1, N-1\}$ and $P_N = [\frac{N-1}{N}, 1]$. Then $\mu(P_i) = \frac{1}{N}$, $i \in \{1, 2, ..., N\}$. Under the uniform partition, the step functions can be represented by the pixel diagram on the unit square.

D. Graphons as Operators

Following [12], a graphon $\mathbf{W} \in \mathbf{G_1^{sp}}$ can be interpreted as an operator $\mathbf{W} : L^2[0,1] \to L^2[0,1]$. The operation on $\mathbf{v} \in L^2[0,1]$ is defined as follows:

$$[\mathbf{W}\mathbf{v}](x) = \int_0^1 \mathbf{W}(x,\alpha)\mathbf{v}(\alpha)d\alpha. \tag{7}$$

The operator product is then defined by

$$[\mathbf{U}\mathbf{W}](x,y) = \int_0^1 \mathbf{U}(x,z)\mathbf{W}(z,y)dz, \tag{8}$$

where $\mathbf{U}, \mathbf{W} \in \mathbf{G_1^{sp}}$. For simplicity of notation, $\mathbf{UW}$ is used to denote the graphon given by the convolution in (8); similarly, $\mathbf{Wv}$ denotes the function defined by (7). Note that if $\mathbf{U} \in \mathbf{G_1^{sp}}$ and $\mathbf{W} \in \mathbf{G_1^{sp}}$, then $\mathbf{UW} \in \mathbf{G_1^{sp}}$, since for all $x, y \in [0,1]$

$$|[\mathbf{U}\mathbf{W}](x,y)| \le \int_0^1 |\mathbf{U}(x,z)\mathbf{W}(z,y)|dz \le 1. \tag{9}$$

Consequently, the power $\mathbf{W}^n$ of an operator $\mathbf{W} \in \mathbf{G_1^{sp}}$ is defined as

$$\mathbf{W}^n(x,y) = \int_{[0,1]^n} \mathbf{W}(x,\alpha_1)\cdots\mathbf{W}(\alpha_{n-1},y)d\alpha_1\cdots d\alpha_{n-1}$$

with $\mathbf{W}^n \in \mathbf{G_1^{sp}}$ $(n \ge 1)$. $\mathbf{W^0}$ is formally defined as the identity operator on functions in $L^2[0,1]$, but we note that $\mathbf{W^0}$ is not a graphon.

E. The Graphon Unitary Operator Algebra

It is evident that the operator composition defined in (8) above yields an operator algebra with a multiplicative binary operation possessing the associativity, left distributivity, right distributivity properties and compatibility with the scalar field $\mathcal{R}$. By adjoining the identity element $\mathbf{I}$ to the algebra $\mathcal{G_A}$ (see e.g. [15]) we obtain a unitary algebra $\mathcal{G_{AI}}$. The identity element $\mathbf{I}$ is defined as follows: for any $\mathbf{W} \in L^2[0,1]^2$

$$[\mathbf{I}\mathbf{W}](x,y) = \int_0^1 \mathbf{W}(z,y)\delta(x,z)dz = \mathbf{W}(x,y), \tag{10}$$

where $\delta(\cdot,z)dz$ is the measure satisfying $\int_0^1 u(z)\delta(x,z)dz = u(x)$ for all $u \in L^2[0,1]$, and in particular $\int_0^1 \delta(x,z)dz = 1$. The graphon unitary operator algebra $\mathcal{G_{AI}}$ will be used in the definition of the controllability Gramian and the input operator. More specifically, we use the subset $\mathcal{G}^1_{\mathcal{AI}} = \{(a\mathbf{I} + \mathbf{A}) : \mathbf{A} \in \mathcal{G}^1_\mathcal{A}, a \in \mathcal{R}\}$ where $\mathcal{G}^1_\mathcal{A}$ is the subset of $\mathcal{G_A}$ that corresponds to $\tilde{\mathbf{G}}_\mathbf{1}^\mathbf{sp}$.

F. Graphon Differential Equations

Let X be a Banach space. A linear operator $A : D(A) \subset X \to X$ is *closed* if $\{(x, Ax) : x \in D(A)\}$ is closed in the product space $X \times X$ (see [13]). $\mathcal{L}(X)$ denotes the Banach algebra of all linear continuous mappings $T : X \to X$. $L^p([a,b];X)$ denotes the Banach space of equivalent classes of strongly measurable (in the Böchner sense) mappings $[a,b] \to X$ that are p-integrable, $1 \le p < \infty$, with norm $\|f\|_{L^p([a,b];X)} = \left[\int_a^b |f(s)|^p ds\right]^{\frac{1}{p}}$. Let $\mathbf{A} : [0,1]^2 \to [-1,1]$ be a graphon and hence a bounded and closed linear operator from $L^2[0,1]$ to $L^2[0,1]$. Following [16], $\mathbf{A}$ is the infinitesimal generator of the uniformly (hence strongly) continuous semigroup $S_\mathbf{A}(t) := e^{\mathbf{A}t} = \sum_{k=0}^\infty \frac{t^k\mathbf{A}^k}{k!}$. Therefore, the initial value problem of the graphon differential equation

$$\dot{\mathbf{y}}_\mathbf{t} = \mathbf{A}\mathbf{y}_\mathbf{t}, \quad \mathbf{y_0} \in L^2[0,1] \tag{11}$$

has a solution given by $\mathbf{y_t} = e^{\mathbf{A}t}\mathbf{y_0}$.

Theorem 3 ([17]). Let $\{\mathbf{A_N}\}_{N=1}^{\infty}$ be a sequence of graphons such that $\mathbf{A_N} \to \mathbf{A}_*$ as $N \to \infty$ in the L^2 metric. Then for all $\mathbf{x} \in L^2[0,1]$, $e^{\mathbf{A_N}t}\mathbf{x} \to e^{\mathbf{A}_*t}\mathbf{x}$ as $N \to \infty$ in the L^2 metric where the convergence is pointwise in time and uniform on any time interval $[0,T]$.

III. Network Systems and Their Limit Systems

A. Scaled Network Systems with Node Averaging Dynamics

Consider an interlinked network S^N of linear (symmetric) dynamical subsystems $\{S_i^N; 1 \le i \le N\}$, each with an n dimensional state space. The subsystem S_i^N at the node V_i in the network $G_N(V,E)$ has interactions with $S_j^N, 1 \le j \le N$, specified as below:

$$S_i^N: \quad \begin{aligned} &\dot{x}_t^i = \frac{1}{nN}\sum_{j=1}^{N} A_{ij}x_t^j + \frac{1}{nN}\sum_{j=1}^{N} B_{ij}u_t^j, \\ &x_t^i, u_t^i \in \mathcal{R}^n, i \in \{1,...,N\}, \end{aligned}$$

with $A_N = [A_{ij}], B_N = [B_{ij}] \in \mathcal{R}^{nN\times nN}$, the (symmetric) block-wise adjacency matrices of $G_N(V,E)$ and of the input graph, where $A_{ij} = [0]$ if S_i^N has no connection to S_j^N and similarity for B_{ij}. Then the (symmetric) linear dynamics for the network system $S^N(A_N, B_N, G_N)$ can be described by

$$S^N: \quad \begin{aligned} &\dot{x}_t = A_N \circ x_t + B_N \circ u_t, \\ &x_t, u_t \in \mathcal{R}^{nN}, \quad A_N, B_N \in \mathcal{R}^{nN\times nN}, \end{aligned} \tag{12}$$

where $\circ$ denotes the so called averaging operator given by $A_N \circ x = \frac{1}{(nN)}A_N x$. Let $\mathcal{S} = \times_{N=1}^{\infty}\mathcal{S^N}$ where $\mathcal{S^N} = \cup_{A_N,B_N,G_N} S^N(A_N,B_N,G_N)$. For simplicity, we require the elements of A_N and B_N to be in $[-1,1]$ for each N (note that in general A_N and B_N have elements that are bounded real numbers for which case we would achieve similar results). In addition, we note that if we take the supremum norm on vectors in $\mathcal{R}^{nN}$, i.e. $\|x\|_\infty = \sup_i |x_i|$, and the corresponding $\circ$ operator norm of A, i.e. $\|A\|_{op} = \sup_{\|x\|_\infty \neq 0} \frac{\|A\circ x\|_\infty}{\|x\|_\infty}$, then $\|A\|_{op} \le 1$.

B. Network Systems with Node Averaging Dynamics Described by Step Functions in the Graphon Space

Let $\{(A_N; B_N)\}_{N=1}^{\infty} \in \mathcal{S}$ be a sequence of systems with the node averaging dynamics each of which is described according to (12). Let $|A_{Nij}| \le 1$ and $|B_{Nij}| \le 1$ for all $i,j \in \{1,...,nN\}$. Let $\mathbf{A_s^{[N]}}, \mathbf{B_s^{[N]}} \in \mathbf{G_1^{sp}}$ be the step functions corresponding one-to-one to A_N and B_N; these are specified using the uniform partition P^{nN} of $[0,1]$ by the following *matrix to step function mapping* M_G: for all $i,j \in \{1,2,...,nN\}$,

$$\mathbf{A_s^{[N]}}(x,y) := A_{Nij}, \quad \forall (x,y) \in P_i \times P_j, \tag{13}$$

and similar for $\mathbf{B_s^{[N]}}$.

Define a *piece-wise constant (PWC) function* on $\mathcal{R}$ to be any function of the form $\sum_{k=1}^{l}\alpha_k\psi_{I_k}$ where $\alpha_1,...,\alpha_l$ are complex numbers and each I_k is a bounded interval (open, closed, or half-open). Let $L^2_{pwc}[0,1]$ denote the space of piece-wise constant $L^2[0,1]$ functions under the uniform partition P^{nN}. Let $\mathbf{u_t^s} \in L^2_{pwc}[0,1]$ correspond one-to-one to $u_t \in \mathcal{R}^{nN}$ via the following *vector to PWC function mapping* also denoted by M_G: for all $i \in \{1,...,nN\}$,

$$\mathbf{u_t^s}(\alpha) := u_t(i), \quad \forall \alpha \in P_i; \tag{14}$$

and similarly $\mathbf{x_t^s} \in L^2_{pwc}[0,1]$ corresponds to $x_t \in \mathcal{R}^{nN}$.

Lemma 1 ([17]). The trajectories of the system in (12) correspond one-to-one under the mapping M_G to the trajectories of the system

$$\begin{aligned} &\dot{\mathbf{x}}_\mathbf{t}^\mathbf{s} = \mathbf{A_s^{[N]}}\mathbf{x_t^s} + \mathbf{B_s^{[N]}}\mathbf{u_t^s}, \\ &\mathbf{x_t^s}, \mathbf{u_t^s} \in L^2_{pwc}[0,1], \mathbf{A_s^{[N]}}, \mathbf{B_s^{[N]}} \in \mathbf{G_1^{sp}} \subset \mathcal{G}^1_{\mathcal{AI}} \end{aligned} \tag{15}$$

with graphon operations defined according to (7).

C. Limits of Sequences of Network Systems

Now the sequence of network systems with the node averaging dynamics can be described by the sequence of step function operators as $\{(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})\}_{N=1}^{\infty}$. Let the graphon sequences $\{\mathbf{A_s^{[N]}}\}$ and $\{\mathbf{B_s^{[N]}}\}$ be Cauchy sequences of step functions in $L^2[0,1]^2$. Due to the completeness of $L^2[0,1]^2$, the respective graphon limits $\mathbf{A}$ and $\mathbf{B}$ exist and these will then necessarily be the limits in the cut metric (see [12]). In fact, we can generalized the control input operator $\mathbb{B}$ to $\mathcal{G}^1_{\mathcal{AI}}$, i.e., $\mathbb{B}$ can consists of the identity operator part and the graphon part as $\mathbb{B} = \beta\mathbf{I} + \mathbf{B}$.

Consider a sequence of systems $\{(\mathbf{A_s^{[N]}}; \mathbb{B}^{[\mathbf{N}]}) \in \tilde{\mathbf{G}}_\mathbf{1}^\mathbf{sp} \times \mathcal{G}^1_{\mathcal{AI}}\}_{N=1}^{\infty}$. Decompose the input operator into the identity part and the graphon part as $\mathbb{B}^{[\mathbf{N}]} = \beta_N\mathbf{I} + \mathbf{B_s^{[N]}}$.

Definition 1. A sequence of systems $\{(\mathbf{A_s^{[N]}}; \mathbb{B}^{[\mathbf{N}]}) \in \tilde{\mathbf{G}}_\mathbf{1}^\mathbf{sp} \times \mathcal{G}^1_{\mathcal{AI}}\}_{N=1}^{\infty}$ is convergent if

1) there exist $\beta \in \mathcal{R}$ such that $\lim_{N\to\infty}\beta_N = \beta$
2) there exist $\mathbf{A}, \mathbf{B} \in \tilde{\mathbf{G}}_\mathbf{1}^\mathbf{sp}$ such that $\{(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})\}$ converges to $(\mathbf{A}; \mathbf{B})$ in the L^2 metric, i.e. $\mathbf{A_s^{[N]}} \to \mathbf{A}$ and $\mathbf{B_s^{[N]}} \to \mathbf{B}$ under the same sequence of measure preserving bijections in the L^2 metric.

Then the limit system is represented by $(\mathbf{A}; \mathbb{B})$ where $\mathbb{B} = \beta\mathbf{I} + \mathbf{B}$. With an abuse of notation, in the following sections we use $\mathbf{B}$ and $\mathbf{B_s^{[N]}}$ to represent input operators in $\mathcal{G}^1_{\mathcal{AI}}$.

IV. The Limit Graphon System and Its Properties

A. Infinite Dimensional Graphon Systems

We follow [13] and specialize the Hilbert space of states H and the Hilbert space of controls U appearing there to the space $L^2(\mathcal{R}; L^2[0,1])$. We formulate an infinite dimensional linear system as follows:

$$LS^\infty: \quad \dot{\mathbf{x}}_\mathbf{t} = \mathbf{A}\mathbf{x_t} + \mathbf{B}\mathbf{u_t}, \quad \mathbf{x_0} \in L^2[0,1], \tag{16}$$

where $\mathbf{A} \in \mathbf{G_1^{sp}}$, $\mathbf{B} \in \mathcal{G}^1_{\mathcal{AI}}$, and hence bounded operators on $L^2[0,1]$, $\mathbf{x_t} \in L^2[0,1]$ is the system state at time t and $\mathbf{u_t} \in L^2[0,1]$ is the control input at time t.

B. Uniqueness of the Solution

A solution $\mathbf{x}_{(\cdot)} \in L^2(\mathcal{R}; L^2[0,1])$ is a *(mild) solution* of (16) if $\mathbf{x_t} = e^{(t-a)\mathbf{A}}\mathbf{x_a} + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}\mathbf{u_s} ds$ for all a and t in $\mathcal{R}$ such that $a \leq t$. Following [13] the assumptions on the operators $\mathbf{A}$ and $\mathbf{B}$ are

$$(H1) \quad \begin{cases} \text{(i)} \quad \mathbf{A} \text{ generates a strongly continuous} \\ \qquad \text{semigroup } e^{t\mathbf{A}} \text{ on } L^2[0,1], \\ \text{(ii)} \quad \mathbf{B} \in \mathcal{L}(L^2[0,1]). \end{cases}$$

Under assumption $(H1)$, the system (16) has a unique solution $\mathbf{x} \in C([0,T]; L^2[0,1])$ for any $\mathbf{x_0} \in L^2[0,1]$ and any $\mathbf{u} \in L^2([0,T]; L^2[0,1])$.

Theorem 4 ([1]). The graphon system LS^∞ in Eq. (16) has a unique solution $\mathbf{x} \in C([0,T]; L^2[0,1])$ for any $\mathbf{x_0} \in L^2[0,1]$ and any $\mathbf{u} \in L^2([0,T]; L^2[0,1])$.

C. Controllability

A system $(\mathbf{A}; \mathbf{B})$ is *exactly controllable* on $[0,T]$ if for any initial state $\mathbf{x}_0 \in L^2[0,1]$ and any target state $\mathbf{x}_f \in L^2[0,1]$, there exists a control $\mathbf{u} \in L^2([0,T]; U)$ driving the system from $\mathbf{x}_0$ to $\mathbf{x}_f$, i.e. $\mathbf{x}_T = \mathbf{x}_f$ with $\mathbf{x}_T = e^{\mathbf{A}T}\mathbf{x_0} + \int_0^T e^{\mathbf{A}(T-t)}\mathbf{B}\mathbf{u_t} dt$. A system $(\mathbf{A}; \mathbf{B})$ is *approximately controllable* on $[0,T]$ if for any initial state $\mathbf{x}_0 \in L^2[0,1]$, any target state $\mathbf{x}_f \in L^2[0,1]$ and any $\varepsilon > 0$, there exists a control $\mathbf{u} \in L^2([0,T]; U)$ driving the system from $\mathbf{x}_0$ to points in the state space within a ε-distance from $\mathbf{x}_f$, i.e. $\|\mathbf{x}_T - \mathbf{x}_f\|_2 \leq \varepsilon$. The *controllability Gramian operator* $\mathbf{W_t} : L^2[0,1] \to L^2[0,1]$ is defined as

$$\mathbf{W_t} := \int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t-s)} ds, \quad t > 0.$$

A necessary and sufficient condition for exact controllability on $[0,T]$ is the uniform positive definiteness of $\mathbf{W}_T$, that is, $\langle \mathbf{W}_T h, h \rangle \geq c_T \|h\|^2$ for all $h \in L^2[0,1]$ where $c_T > 0$ and $\|\cdot\|$ is the $L^2[0,1]$ norm (see [13], [14]). The positive definiteness of the controllability Gramian operator $\mathbf{W}_T$ is equivalent to the approximate controllability of the corresponding system (see [13], [14]).

Theorem 5 ([17]). Let $\mathbf{A}$ be a graphon in $\tilde{\mathbf{G}}_\mathbf{1}^{\mathbf{sp}}$ and let $\mathbf{B}$ be a bounded linear operator on $L^2[0,1]$. The linear system $(\mathbf{A}; \mathbf{B})$ is exactly controllable on a finite time horizon $[0,T]$ if all the values in the spectrum of $\mathbf{B}\mathbf{B}^T$ are lower bounded by a strictly positive constant.

Proposition 1 ([17]). Let $\mathbf{A}$ and $\mathbf{B}$ be graphons in $\tilde{\mathbf{G}}_\mathbf{1}^{\mathbf{sp}}$. Then $(\mathbf{A}; \mathbf{B})$ is not exactly controllable on any finite time horizon $[0,T]$.

V. Graphon Linear Quadratic Regulation of Network Systems

A. LQR Problems for Graphon Dynamical Systems

Let $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the norm and the inner product in $L^2[0,1]$. For finite $T > 0$, consider the problem of minimizing the cost given by

$$J(\mathbf{u}) = \int_0^T \left[\|\mathbf{C}\mathbf{x}_\tau\|^2 + \|\mathbf{u}_\tau\|^2\right] d\tau + \langle \mathbf{P}_0\mathbf{x}_T, \mathbf{x}_T \rangle \quad (17)$$

over all controls $\mathbf{u} \in L^2([0,T]; L^2(0,1))$ subject to the system model constrains in (16). The assumptions for $\mathbf{C}$ and $\mathbf{P}_0$ are:

$$(H2) \quad \begin{cases} \text{(iii)} \quad \mathbf{P}_0 \in \mathcal{L}(L^2[0,1]) \text{ is hermitian and} \\ \qquad \text{non-negative,} \\ \text{(iv)} \quad \mathbf{C} \in \mathcal{L}(L^2[0,1]). \end{cases}$$

B. Existence and Uniqueness of Solutions to LQR Problems

Finding the feedback control via dynamic programming consists of the two following steps:

Step 1. Solve the Riccati equation

$$\dot{\mathbf{P}} = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{C}, \quad \mathbf{P}(0) = \mathbf{P}_0 \quad (18)$$

Step 2. Given the solution $\mathbf{P}$ to the Riccati equation, it can be proved that the optimal control $\mathbf{u}^*$ is given by

$$\mathbf{u}_t^* = -\mathbf{B}^T\mathbf{P}(T-t)\mathbf{x}_t^*, \quad t \in [0,T] \quad (19)$$

and moreover that $\mathbf{x}^*$ is the solution of the closed loop equation

$$\begin{aligned} &\dot{\mathbf{x}}_t = \mathbf{A}\mathbf{x}_t - \mathbf{B}\mathbf{B}^T\mathbf{P}(T-t)\mathbf{x}_t, \\ &t \in [0,T], \mathbf{x}_0 \in L^2[0,1]. \end{aligned} \quad (20)$$

Applying the results in [13] to $L^2[0,1]$ space, we establish the existence and uniqueness of the solution to the Riccati equation (18) and the existence and uniqueness of optimal solution pair $(\mathbf{u}^*, \mathbf{x}^*)$ in (19) and (20) under the assumptions $(H1)$ and $(H2)$.

C. The Graphon-Network LQR (GLQR) Strategy

Consider the control problem of regulating the states of each member of $\{(A_N; B_N)\}_{N=1}^\infty \in \mathcal{S}$.

The **Graphon-Network LQR (GLQR) Strategy** is then:

S.1 *Let* $\{(\mathbf{A}_\mathbf{s}^{[\mathbf{N}]}; \mathbf{B}_\mathbf{s}^{[\mathbf{N}]}) \in \tilde{\mathbf{G}}_\mathbf{1}^{\mathbf{sp}} \times \mathcal{G}_{\mathcal{AI}}^1\}_{N=1}^\infty$ *be the sequence of step function systems equivalent to* $\{(A_N; B_N)\}_{N=1}^\infty \in \mathcal{S}$ *under the mapping* M_G *and assume that it converges to the graphon system* $(\mathbf{A}; \mathbf{B}) \in \tilde{\mathbf{G}}_\mathbf{1}^{\mathbf{sp}} \times \mathcal{G}_{\mathcal{AI}}^1$.

S.2 *Define the linear quadratic cost for* $(\mathbf{A}; \mathbf{B})$ *as*

$$J(\mathbf{u}) = \int_0^T [\|\mathbf{C}\mathbf{x}_\tau\|^2 + \|\mathbf{u}_\tau\|^2] d\tau + \langle \mathbf{P}_0\mathbf{x}_T, \mathbf{x}_T \rangle$$

and the linear quadratic cost for $(\mathbf{A}_\mathbf{s}^{[\mathbf{N}]}; \mathbf{B}_\mathbf{s}^{[\mathbf{N}]})$ *as*

$$J(\mathbf{u}^{[\mathbf{N}]}) = \int_0^T [\|\mathbf{C}_\mathbf{s}^{[\mathbf{N}]}\mathbf{x}_\mathbf{t}^{[\mathbf{N}]}\|^2 + \|\mathbf{u}_\mathbf{t}^{[\mathbf{N}]}\|^2] dt + \langle \mathbf{P}_\mathbf{s0}^{[\mathbf{N}]}\mathbf{x}_\mathbf{T}^{[\mathbf{N}]}, \mathbf{x}_\mathbf{T}^{[\mathbf{N}]} \rangle$$

where it is assumed that $\mathbf{C}_\mathbf{s}^{[\mathbf{N}]} \to \mathbf{C}$ *and* $\mathbf{P}_\mathbf{s0}^{[\mathbf{N}]} \to \mathbf{P}$ *in the strong operator sense. Solve the infinite dimensional Riccati equation for* $(\mathbf{A}; \mathbf{B})$ *to generate the solution* $\mathbf{P}$.

S.3 *Approximate* $\mathbf{P}$ *to generate* $\tilde{\mathbf{P}}_\mathbf{N}$ *and hence the control law* $\mathbf{u}_\mathbf{t}^{[\mathbf{N}]} = -\mathbf{B}_\mathbf{s}^{[\mathbf{N}]T}\tilde{\mathbf{P}}_\mathbf{N}(T-t)\mathbf{x}_\mathbf{t}^{[\mathbf{N}]}$ *for* $(\mathbf{A}_\mathbf{s}^{[\mathbf{N}]}; \mathbf{B}_\mathbf{s}^{[\mathbf{N}]})$.

In this work, the basic assumption in the formulation of LQR problems for linear systems distributed on complex networks is that the regulation problem for the infinite dimensional graphon limit systems can be solved (e.g. by established approximation methods or through methods based on spectral decompositions [18]) while the finite dimensional LQR problems for the original complex network systems are intractable due to their cardinality.

D. Control Law Approximations

By approximating the Riccati equation solution $\mathbf{P}$ for $(\mathbf{A};\mathbf{B})$ we can generate $\tilde{\mathbf{P}}_{\mathbf{N}}$ that provides the control law for the finite dimensional network system.

$$\mathbf{u}_{\mathbf{t}}^{[\mathbf{N}]} = -{\mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]}}^T \tilde{\mathbf{P}}_{\mathbf{N}}(T-t)\mathbf{x}_{\mathbf{t}}^{[\mathbf{N}]}.$$

Let $\Sigma(L^2[0,1])) = \{T \in \mathcal{L}(L^2[0,1])) : T \text{ is Hermition}\}$ and

$$\begin{aligned}&\Sigma^+(L^2[0,1])\\ &\quad = \{T \in \Sigma(L^2[0,1]) : \langle Tx, x\rangle \geq 0, \forall x \in L^2[0,1]\}.\end{aligned}$$

Denote the topological space of all strongly continuous mappings $F : I \to \Sigma(L^2[0,1])$ endowed with strong convergence (see [13]) by $C_s(I; \Sigma(L^2[0,1]))$.

1) Approximation of the Solution to the Riccati Equation: First, we construct the equivalent representation of the linear operator $\mathbf{P}$ in $C_s([0,T]; \Sigma^+(L^2[0,1]))$ by integration against measures, that is, we first represent $\mathbf{P}$ by

$$\mathbf{P}(\cdot)(x,y)d\sigma(x,y), \quad (x,y) \in [0,1]^2, \tag{21}$$

where $\sigma(x,y)$ represents the measure (which can be a singular measure, a Lebesgue measure or a mixed measure).

Second, we introduce a method to approximate the operator $\mathbf{P}$ by local integration with respect to measures over partitions. The local *step function approximation against measures* $\tilde{\mathbf{P}}_{\mathbf{N}}$ *of* $\mathbf{P}$ is defined by integration against measures as follows: for $(x,y) \in S_i \times S_j$ with $S_i, S_j \subset [0,1]$ representing the elements of the partition,

$$\tilde{\mathbf{P}}_{\mathbf{N}}(\cdot)(x,y) = \frac{\int_{S_i \times S_j} \mathbf{P}(\cdot)(x,y)d\sigma(x,y)}{\mu(S_i) \times \mu(S_j)}, \tag{22}$$

where $\mu(S_i)$ represents the length of the interval S_i and $\sigma(x,y)$ represents the measure (which can be a singular measure, a Lebesgue measure or a mixed measure).

2) Approximation of the Riccati Solution and Its Convergence to the Optimal Riccati Solution: Based on the definition of the step function approximation against measures, $\tilde{\mathbf{P}}_{\mathbf{N}}(\cdot)x$ is the PWC function approximation of $\mathbf{P}(\cdot)x$ in $L^2[0,1]$, and hence for any $x \in L^2[0,1]$,

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|\tilde{\mathbf{P}}_{\mathbf{N}}(t)x - \mathbf{P}(t)x\|_2 = 0.$$

Therefore we obtain the following lemma.

Lemma 2 ([17]). Let $\tilde{\mathbf{P}}_{\mathbf{N}}$ be generated by step function approximation against measures from $\mathbf{P}$ via an $N \times N$ uniform partition of $[0,1]^2$. Then

$$\lim_{N\to\infty} \tilde{\mathbf{P}}_{\mathbf{N}} = \mathbf{P}, \quad \text{in } C_s([0,T]; \Sigma(L^2[0,1])).$$

Theorem 6 ([17]). Let $\tilde{\mathbf{P}}_{\mathbf{N}}$ be generated by step function approximation against measures from $\mathbf{P}$ via $N \times N$ uniform partition of $[0,1]^2$. For any $x \in L^2[0,1]$, for any $t \in [0,T]$,

$$\lim_{N\to\infty} \|\tilde{\mathbf{P}}_{\mathbf{N}}(t)x - \mathbf{P}_{\mathbf{s}}^{[\mathbf{N}]}(t)x\|_2 = 0,$$

where $\mathbf{P}_{\mathbf{s}}^{[\mathbf{N}]}$ is the solution of Riccati equation of $(\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}; \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]})$ that converges strongly to the solution $\mathbf{P}$.

3) Convergence of States and Convergence of Costs: Let $\mathbf{P}_{\mathbf{s}}^{[\mathbf{N}]}$ denote the solution of the Riccati equation for $(\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}; \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]})$ that converges strongly to the solution $\mathbf{P}$ of the Riccati equation for $(\mathbf{A};\mathbf{B})$. And further let $\tilde{\mathbf{P}}_{\mathbf{N}}$ be the step function approximation against measures for $\mathbf{P}$ generated via the $N \times N$ uniform partition of $[0,1]^2$.

Theorem 7 ([17]). Consider the time horizon $[0,T]$. Let the optimal linear quadratic control law for $(\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}; \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]})$ be generated by

$$\mathbf{u}_t^{N*} = -{\mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]}}^T \mathbf{P}_{\mathbf{s}}^{[\mathbf{N}]}(T-t)\mathbf{x}_t^{N*},$$

where the optimal state trajectory is given by $\mathbf{x}^{N*}$, and let the graphon approximate control law for $(\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}; \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]})$ be

$$\mathbf{u}_{\mathbf{t}}^{[\mathbf{N}]} = -{\mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]}}^T \tilde{\mathbf{P}}_{\mathbf{N}}(T-t)\mathbf{x}_{\mathbf{t}}^{[\mathbf{N}]},$$

where the corresponding state trajectory is given by $\mathbf{x}^{[\mathbf{N}]}$. Then for all $t \in [0,T]$, $\lim_{N\to\infty} \|\mathbf{x}_t^{N*} - \mathbf{x}_{\mathbf{t}}^{[\mathbf{N}]}\|_2 = 0$, and $\lim_{N\to\infty} |J(\mathbf{u}^{N*}) - J(\mathbf{u}^{[\mathbf{N}]})| = 0$.

Note that $J(\mathbf{u}^{[\mathbf{N}]})$ is not guaranteed to be minimal, since it is the cost under the approximate control. However, this result shows that $J(\mathbf{u}^{[\mathbf{N}]})$ is guaranteed to converge to the optimal cost as the size of the network increases.

VI. Simulation Example

Consider a network system evolving according to the node averaging dynamics on a weighted graph G_N. Suppose each node has an independent input. Denote the system by $(A_N; I_N)$, where A_N is the adjacency matrix of G_N and I_N is the identity input mapping. The network system $(A_N; I_N)$ with (normalized) node dynamics is therefore described by

$$\dot{x}_t^i = \frac{1}{N}\sum_{j=1}^{N} A_{Nij} x_t^j + u_t^i, \quad x_t^i, u_t^i \in \mathcal{R}, i \in \{1, ..., N\}. \tag{23}$$

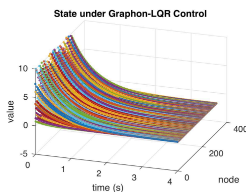
The regulation objective is to regulate the network states around origin from random initial states with minimum quadratic cost.

As an example, we consider a sequence of networks converging to the graphon limit $\mathbf{U}(x,y) = 4\cos(2\pi(x-y))$, $x, y \in [0,1]$, which is depicted in Figure 2(h), and we solve the LQR problem over the time horizon $[0,T]$ for the network sequence. (See [1] for a detailed description of the generation of a convergent network sequence). In this simulation, as shown in Figure 2, a network of size 320 in the sequence is considered. The system is represented by (A_{320}, I_{320}), with A_{320} as the adjacency matrix of the weighted network and I_{320} as the identity input matrix of size 320. $B = I_{320}$, $C = \sqrt{2}I_{320}$, $P_0 = I_{320}$.

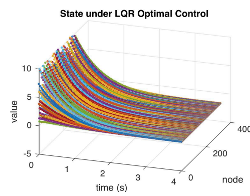
The infinite dimensional limit Riccati equation can be solved with the solution given by $\mathbf{P}_t = \alpha_t \mathbf{I} + \beta_t \mathbf{U}$, where α_t and β_t satisfy

$$\dot{\alpha}_t = 2 - \alpha_t^2, \quad \dot{\beta}_t = 2\alpha_t + 16\beta_t - 2\alpha_t\beta_t - 8\beta_t^2,$$

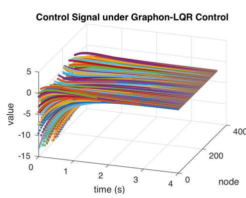
with $\alpha_0 = 1, \beta_0 = 0$. The finite dimensional control law is then generated by approximating the Riccati equation solution as in (22). As the networks increase in size and converge to the limit graphon, the strong convergence of the approximated graphon Riccati equation solution to the finite dimensional Riccati equation solution is guaranteed by Theorem 6. Furthermore, the convergence in the state trajectory (and cost) to the optimal state trajectory (and the optimal cost) is guaranteed by Theorem 7. Both the



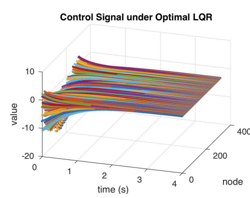
(a) State Evolution under Graphon Control



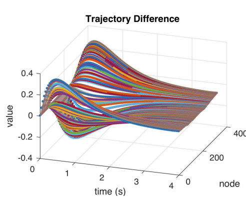
(b) State Evolution under Optimal LQR



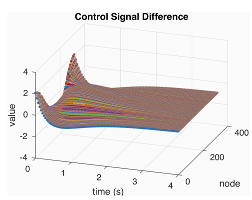
(c) Control Input of Graphon Control



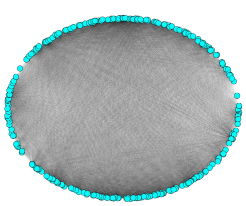
(d) Control Input of Optimal LQR



(e) State Difference



(f) Control Signal Difference


(g) Network of 320 Nodes

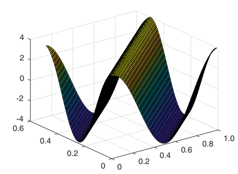

(h) Graphon Limit

Fig. 2. Simulation on a Network of 320 Nodes

graphon-LQR control and the LQR optimal control regulate the system to the origin from the same random initial states as shown in Figures 2(a) and 2(b). Figures 2(e) and 2(f) depict a remarkably similar performance of the approximated graphon-LQR control and the LQR optimal control for the 320 node system. The maximum trajectory difference from the optimal control is less than 4% of the maximum initial states. With the graph interpreted as an $L^2[0,1]^2$ function, the distance between the graph and the graphon limit in $L^2[0,1]^2$ is 0.000813, and the graphon-LQR control cost is only 0.133% higher than the optimal LQR control cost.

VII. Conclusion

Important aspects of the theory introduced in this paper which require further research studies include: (1) the application of the regulation strategy to asymmetric (i.e., directed) network systems; (2) an equivalent theory for sparse networks; (3) the application of graphon control to linear quadratic tracking problems on networks; (4) the application of graphon control to stochastic linear quadratic Gaussian problems; (5) the graphon control of networks of nonlinear systems. Finally, this paper only deals with centralized control, however, building upon the approach developed here, the decentralized control of complex systems is formulated within a graphon theoretic mean field game framework in [19] and a team theoretic optimal control framework in [18].

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