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LQG Graphon Mean Field Games: Graphon Invariant Subspaces

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Abstract— **This paper studies approximate solutions to large-scale linear quadratic stochastic games with homogeneous nodal dynamics and heterogeneous network couplings based on the graphon mean field game framework in [1]–[3]. A graphon time-varying dynamical system model is first formulated to study the limit problem of linear quadratic Gaussian graphon mean field games (LQG-GMFG). The Nash equilibrium to the limit problem is then characterized by two coupled graphon time-varying dynamical systems. Based on this characterization, we establish two sufficient conditions for the existence of a unique solution to the limit LQG-GMFG problem, and moreover we provide a new asymptotic error bound for applications of approximate solutions to finite-network games. Finally, simulation results on random networks are demonstrated**

I. Introduction

Applications such as market networks, large-scale social networks, advertising networks, communication networks and smart grids involve strategic decisions over a large number of agents coupled via large-scale heterogeneous network structures. The large cardinalities of the underlying networks and the complexity of the underlying network couplings in dynamics and decision strategies make such problems challenging or even intractable by standard methods. To characterize large graphs and study the convergence of dense graph sequences to their limits, graphon theory is established in [4]–[6]. It has been applied to study dynamical systems ([7], [8]), network centrality [9], random walks [10], graph neural networks [11], epidemic models ([12], [13]), Graphon Control of very large-scale networks ([14]–[17]), among others. To study strategic decision problems on networks, game theoretic models with various interpretations of the underlying networks have been extensively studied in the literature (see for instance [18]–[21]). To model and solve game problems on large-scale non-uniform networks, static and dynamic games on graphons are studied ([22], [23], [1]–[3], [24]). In particular, for dynamic game problems involving large populations of individuals on non-uniform networks, Graphon Mean Field Game (GMFG) theory was proposed and developed in [1]–[3]. It generalizes classical mean field game theory ([25], [26]) in the sense that each node may be influenced by a different local mean field. Under suitable technical conditions, Nash equilibria and ε-Nash properties have been established in [1]–[3]. Mean field games with non-uniform cost couplings were studied in an earlier paper [27], and mean field game problems on graphs with different interpretations of the underlying graphs have also been studied in [28]–[30]. In [28], [29], the graphs represent physical constraints on the state space of the mean field game problems. In [30] linear quadratic mean field games over Erdös-Rényi graphs are studied where the associated asymptotic game is a classical mean field game. Recent works on mean field game problems on networks include [24], [31]. Among these papers, depending on the definitions of nodes, there are two classes of closely related mean field game problems on networks: (i) networks of mean field (or measure) couplings where each node on the network represents a population [1]–[3]; (ii) networks of individual state couplings where each node represents an agent (see for instance [24], [27], [30], [31]). In the current paper, each node represents a population of homogenous agents.

The main contributions of this paper include the following: (i) characterization of the limit LQG-GMFG problem by two coupled graphon time-varying dynamical systems, (ii) two different sufficient conditions on the existence of a unique solution to the limit LQG-GMFG problem, (iii) new asymptotic error bounds on the convergence of the network mean fields to the graphon mean field, and (iv) two solution methods via invariant subspace decompositions for the limit LQG-GMFG problems, one based on fixed point iterations and the other based on a decoupling Riccati equation.

Notation: $\mathbb{R}$ denotes the set of real numbers. Bold face letters (e.g. $\mathbf{A}, \mathbf{u}$) are used to represent graphons, compact operators and functions. Blackboard bold letters (e.g. $\mathbb{A}$) are used to denote linear operators which are not necessarily compact. $\mathbb{A}^\intercal$ to denote the adjoint operator of $\mathbb{A}$. $\mathcal{W}_c$ denotes the set of all symmetric bounded measurable functions $\mathbf{W} : [0,1]^2 \to [-c, c]$ with $c > 0$; $\mathcal{W}_0$ denotes the set of all symmetric measurable functions $\mathbf{W} : [0,1]^2 \to [0,1]$. For a Hilbert space $\mathcal{H}$, $\mathcal{L}(\mathcal{H})$ denotes the Banach algebra of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. $\mathcal{L}(\mathcal{H})$ endowed with the uniform operator topology is denoted by $\mathcal{L}_u(\mathcal{H})$. For a Banach space $\mathcal{X}$, $C([0,T];\mathcal{X})$ denotes the set of all continuous functions from $[0,T]$ to $\mathcal{X}$. Let $\oplus$ denote direct sum. Let $\otimes$ denote matrix Kronecker product. For any matrix $Q \in \mathbb{R}^{n\times n}$, $Q \geq 0$ (resp. $Q > 0$) means $Q^\intercal = Q$ and $x^\intercal Q x \geq 0$ (resp. $x^\intercal Q x > 0$) for all $x \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$ and $Q \geq 0$, let $\|x\|_Q^2 \triangleq x^\intercal Q x$. $L^2[0,1]$ denotes the space of L^2 measurable functions from $[0,1]$ to $\mathbb{R}$. Let $(L^2[0,1])^n \triangleq \underbrace{L^2[0,1]\times\cdots\times L^2[0,1]}_{n}$.

The inner product in $(L^2[0,1])^n$ is defined as follows: for

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$\mathbf{v}, \mathbf{u} \in (L^2[0,1])^n$, $\langle \mathbf{u}, \mathbf{v}\rangle \triangleq \sum_{i=1}^n \int_{[0,1]} \mathbf{v}_i(\alpha)\mathbf{u}_i(\alpha)d\alpha = \int_{[0,1]} \langle \mathbf{v}(\alpha), \mathbf{u}(\alpha)\rangle_{\mathbb{R}^n} d\alpha$, where $\mathbf{u}_i(\cdot) \in L^2[0,1]$ with $i \in \{1, \ldots, n\}$ denotes the ith component of $\mathbf{u}$ and $\mathbf{u}(\alpha) \in \mathbb{R}^n$ denotes the vector associated with index $\alpha \in [0,1]$. The space $(L^2[0,1])^n$ with the above inner product is a Hilbert space with the corresponding norm $\|\mathbf{v}\|_{(L^2[0,1])^n} \triangleq \left(\int_{[0,1]} \|\mathbf{v}(\alpha)\|^2_{\mathbb{R}^n} d\alpha\right)^{\frac{1}{2}}$. We use $L^2([0,T]; (L^2[0,1])^n)$ to denote the Hilbert space of equivalence classes of strongly measurable (in the Böchner sense [32, p.103]) mappings from $[0,T]$ to $(L^2[0,1])^n$ that are integrable with the norm $\|\mathbf{x}\|_{L^2([0,T];(L^2[0,1])^n)} = \left(\int_0^T \|\mathbf{x}(t)\|^2_{(L^2[0,1])^n} dt\right)^{\frac{1}{2}}$. The function $\mathbf{1} \in L^2[0,1]$ is defined as follows: for all $\alpha \in [0,1]$, $\mathbf{1}(\alpha) = 1$. $\mathbf{1}_n$ denotes the n-dimensional vector of ones. For any $v \in \mathbb{R}^n$ and $\mathbf{f} \in L^2[0,1]$, $v\mathbf{f}$ denotes the function in $(L^2[0,1])^n$ such that $(v\mathbf{f})(\alpha) = v\mathbf{f}(\alpha) \in \mathbb{R}^n$ for all $\alpha \in [0,1]$. For any two functions f and g defined on subsets of $\mathbb{R}$, $f = O(g)$ means that there exist a positive real constant c and a number x_0 such that $|f(x)| \leq cg(x)$ holds for all $x \geq x_0$.

II. Preliminaries

A. Graphs, Graphons and Graphon Operators

A graph $G = (V, E)$ is specified by a node set $V = \{1, \ldots, N\}$ and an edge set $E \subset V \times V$. The corresponding adjacency matrix $W = [w_{ij}]$ is defined as follows: $w_{ij} = 1$ if $(i,j) \in E$ otherwise $w_{ij} = 0$. A graph is undirected if its edge pair is unordered. For a weighted undirected graph, w_{ij} in the adjacency matrix represents the weight between nodes i and j. Furthermore an adjacency matrix can be represented by a pixel diagram on the unit square $[0,1]^2 \subset \mathbb{R}^2$, which corresponds to a step function graphon [6].

Graphons are formally defined as symmetric Lebesgue measurable functions $\mathbf{M} : [0,1]^2 \to [0,1]$. In this paper, we consider symmetric Lebesgue measurable functions $\mathbf{M} : [0,1]^2 \to [-c, c]$ with $c > 0$, and the space of all such functions is denoted by $\mathcal{W}_c$. The space $\mathcal{W}_c$ is compact under the cut metric after identifying equivalent points of cut distance zero [6].

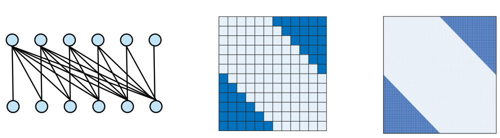

Fig. 1: A half graph, its pixel diagram, and its limit graphon

A graphon $\mathbf{M} \in \mathcal{W}_c$ also defines a self-adjoint bounded linear operator from $L^2[0,1]$ to $L^2[0,1]$ as follows:

$$[\mathbf{M}\mathbf{v}](\alpha) = \int_{[0,1]} \mathbf{M}(\alpha, \eta)\mathbf{v}(\eta)d\eta, \quad \forall \alpha \in [0,1], \tag{1}$$

where $\mathbf{v}, \mathbf{M}\mathbf{v} \in L^2[0,1]$. Following [17], graphons can also be associated with operators from $(L^2[0,1])^n$ to $(L^2[0,1])^n$. Let $\mathcal{L}\left(\left(L^2[0,1]\right)^n\right)$ represent the set of bounded linear operators from $\left(L^2[0,1]\right)^n$ to $\left(L^2[0,1]\right)^n$. Define the operator $[D\mathbf{M}] \in \mathcal{L}\left(\left(L^2[0,1]\right)^n\right)$ with $D \in \mathbb{R}^{n\times n}$ and $\mathbf{M} \in \mathcal{W}_c$ as follows: for any $\mathbf{v} \in \left(L^2[0,1]\right)^n$ and any index $\alpha \in [0,1]$,

$$([D\mathbf{M}]\mathbf{v})(\alpha) \triangleq D \begin{pmatrix} \int_{[0,1]} \mathbf{M}(\alpha,\beta)\mathbf{v}_1(\beta)d\beta \\ \vdots \\ \int_{[0,1]} \mathbf{M}(\alpha,\beta)\mathbf{v}_n(\beta)d\beta \end{pmatrix} \in \mathbb{R}^n. \tag{2}$$

For the identity operator $\mathbb{I}$ in $\mathcal{L}\left(L^2[0,1]\right)$ and $D \in \mathbb{R}^{n\times n}$, the operation $[D\mathbb{I}]$ in $\mathcal{L}\left(\left(L^2[0,1]\right)^n\right)$ is defined as follows: for any $\mathbf{v} \in \left(L^2[0,1]\right)^n$ and any index $\alpha \in [0,1]$,

$$([D\mathbb{I}]\mathbf{v})(\alpha) \triangleq D\left(\mathbf{v}_1(\alpha) \ldots \mathbf{v}_n(\alpha)\right)^{\intercal} = D\mathbf{v}(\alpha) \in \mathbb{R}^n. \tag{3}$$

We use the square bracket $[\cdot]$ to indicate that the operator is in $\mathcal{L}\left(\left(L^2[0,1]\right)^n\right)$. Based on the definitions of the operations of $[D\mathbf{M}]$ and $[D\mathbb{I}]$ in (2) and (3), the kth ($k \geq 0$) power functions of $[D\mathbf{M}]$ and $[D\mathbb{I}]$ are respectively given by $[D\mathbf{M}]^k = [D^k\mathbf{M}^k]$ and $[D\mathbb{I}]^k = [D^k\mathbb{I}^k]$. We note that $\mathbf{M}^0$ is defined as the identity operator from $L^2[0,1]$ to $L^2[0,1]$. Furthermore, for any $A, D \in \mathbb{R}^{n\times n}$, $\mathbb{T}_1, \mathbb{T}_2 \in \{\alpha\mathbb{I} + \beta\mathbf{M} : \alpha, \beta \in \mathbb{R}, \mathbf{M} \in \mathcal{W}_c\} \subset \mathcal{L}((L^2[0,1])^n)$, then $[A\mathbb{T}_1][D\mathbb{T}_2] = [AD\mathbb{T}_1\mathbb{T}_2]$ holds. Since $[D\mathbf{M}]$ is a bounded linear operator from $\left(L^2([0,1])\right)^n$ to $\left(L^2([0,1])\right)^n$, it generates a uniformly continuous (hence strongly continuous) semigroup [33] given by $S_{[D\mathbf{M}]}(t) = \exp(t[D\mathbf{M}]) \triangleq \sum_{k=0}^{\infty} \frac{1}{k!} t^k [D\mathbf{M}]^k$, $t \geq 0$.

B. Invariant Subspace and Component-Wise Decomposition

Let $\mathcal{H}$ denote a Hilbert space. An *invariant subspace* of a bounded linear operator $\mathbb{T} \in \mathcal{L}(\mathcal{H})$ is defined as any subspace $\mathcal{S}_{\mathcal{H}} \subset \mathcal{H}$ such that $\mathbb{T}\mathcal{S}_{\mathcal{H}} \subset \mathcal{S}_{\mathcal{H}}$. Then the subspace $\mathcal{S}_{\mathcal{H}}$ is $\mathbb{T}$*-invariant*. Since a graphon $\mathbf{M} \in \mathcal{W}_c$ defines a self-adjoint operator as in (1), for any invariant subspace $\mathcal{S} \subset L^2[0,1]$ of $\mathbf{M}$, $\mathcal{S}^\perp$ is also an invariant subspace of $\mathbf{M}$ (see [17]) where $S^\perp$ denotes the orthogonal complement subspace of $\mathcal{S}$ in $L^2[0,1]$. A subspace $\mathcal{S} \subset L^2[0,1]$ is the *graphon invariant subspace* of $\mathbf{M} \in \mathcal{W}_c$ if (i) $\mathbf{M}\mathcal{S} \subset \mathcal{S}$, (ii) $\mathbf{M}\mathbf{v} \neq 0$ for all $\mathbf{v} \in (\mathcal{S}\backslash\{0\})$, and (iii) $\mathbf{M}\mathcal{S}^\perp = \{0\}$, where it is readily verified that such a subspace $\mathcal{S}$ is unique. Let $(\mathcal{S})^n \triangleq \underbrace{\mathcal{S} \times \ldots \times \mathcal{S}}_{n} \subset (L^2[0,1])^n$. Clearly, by definition, $(\mathcal{S} \oplus \mathcal{S}^\perp)^n = (L^2[0,1])^n$. Any $\mathbf{v} \in (L^2[0,1])^n$ can be uniquely decomposed through its components as

$$\mathbf{v}_i = \bar{\mathbf{v}}_i + \mathbf{v}_i^\perp, \quad \forall i \in \{1, ..., n\} \tag{4}$$

where $\bar{\mathbf{v}}_i \in \mathcal{S} \subset L^2[0,1]$ and $\mathbf{v}_i^\perp \in \mathcal{S}^\perp \subset L^2[0,1]$. We call this decomposition in (4) the *component-wise decomposition* of $\mathbf{v}$ into $(\mathcal{S})^n$ and $(\mathcal{S}^\perp)^n$, and denote it by $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}^\perp$ where $\bar{\mathbf{v}} \in (\mathcal{S})^n$ and $\mathbf{v}^\perp \in (\mathcal{S}^\perp)^n$ (see [17] for more details).

III. Graphon Dynamical Systems

A. Graphon Dynamical System Models

Consider the graphon time-varying dynamical system

$$\dot{\mathbf{x}}(t) = [A(t)\mathbb{I} + D(t)\mathbf{M}]\mathbf{x}(t) + [B(t)\mathbb{I} + E(t)\mathbf{M}]\mathbf{u}(t) \tag{5}$$

where $\mathbf{x}(t), \mathbf{u}(t) \in \left(L^2[0,1]\right)^n$ and $\mathbf{M} \in \mathcal{W}_c$. The admissible control $\mathbf{u}(\cdot)$ lies in $L^2([0,T]; (L^2[0,1])^n)$. For any $t \in [0,T]$, $A(t)$, $B(t)$, $D(t)$ and $E(t)$ are $n \times n$ matrices;

furthermore, $A(\cdot)$, $B(\cdot)$, $D(\cdot)$ and $E(\cdot)$ are assumed to be continuous from $[0,T]$ to $\mathbb{R}^{n\times n}$.

A *mild solution* of (5) is defined as the solution $\mathbf{x}$ that is continuous in $[0,T]$ and verifies the integral equation $\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t (\mathbb{A}(\tau)\mathbf{x}(\tau) + \mathbb{B}(\tau)\mathbf{u}(\tau))\, d\tau$, where $\mathbb{A}(t) = [A(t)\mathbb{I} + D(t)\mathbf{M}]\mathbf{x}(t)$ and $\mathbb{B}(t) = [B(t)\mathbb{I} + E(t)\mathbf{M}]\mathbf{u}(t)$.

Lemma 1 ([34]) *The system* (5) *has a unique mild solution in* $C([0,T];(L^2[0,1])^n)$ *given by*

$$\mathbf{x}(t) = \Phi(t,0)\mathbf{x}(0) + \int_0^t \Phi(t,\tau)[B(\tau)\mathbb{I} + D(\tau)\mathbf{M}]\mathbf{u}(\tau)d\tau \tag{6}$$

where $\Phi(\cdot,\cdot)$ *is the evolution operator relative to* $[A(\cdot)\mathbb{I} + D(\cdot)\mathbf{M}]$ *that satisfies*

$$\frac{d\Phi(t,\tau)}{dt} = [A(t)\mathbb{I} + D(t)\mathbf{M}]\Phi(t,\tau),\ \Phi(\tau,\tau) = \mathbb{I}, \tag{7}$$

with $\Phi(t,\tau) \in \mathcal{L}_u\big((L^2[0,1])^n\big)$ *for all* $\tau, t \in [0,T]$. *Furthermore, if* $\mathbf{u} \in C([0,T];(L^2[0,1])^n)$*, then* (5) *has a unique classical solution*[1]*, which is also given by* (6). □

Remark 1 Compared to [17], the graphon dynamical system model in (5) is time-varying; more specifically, the parameter matrices $A(\cdot), B(\cdot), D(\cdot)$ and $E(\cdot)$ are time-varying, but the underlying graphon is time-invariant. This time-varying formulation is crucial in enabling the representation of solutions to the limit LQG-GMFG problems via two coupled time-varying graphon differential equations (see Section IV-C). □

B. Graphon Models of Finite Network Systems

Consider an N-node network with the following nodal dynamics: for $i \in \{1,...,N\}$,

$$\dot{x}_i(t) = A(t)x_i(t) + B(t)u_i(t) + D(t)x_i^{\mathcal{G}}(t) + E(t)u_i^{\mathcal{G}}(t) \tag{8}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ represent respectively the state and the control of ith node at time t, and $x_i^{\mathcal{G}}(t) \triangleq \frac{1}{N}\sum_{j=1}^N m_{ij}x_j(t)$ and $u_i^{\mathcal{G}}(t) \triangleq \frac{1}{N}\sum_{j=1}^N m_{ij}u_j(t)$ represent respectively the network influence of states and that of the control at time $t \in [0,T]$. The coupling weights satisfy that $m_{ij} \le c$ for all $i,j \in \{1,...,N\}$ where c is the same constant for the graphon set $\mathcal{W}_c$. We note that problems with m-dimensional control inputs ($m < n$) for the nodal dynamics can be represented by placing zeros in columns (with indices between m and n) of $D(t)$ and $E(t)$. Consider a uniform partition $\{P_1,\dots,P_N\}$ of $[0,1]$ with $P_1 = [0, \frac{1}{N}]$ and $P_k = (\frac{k-1}{N}, \frac{k}{N}]$ for $2 \le k \le N$. The *step function graphon* $\mathbf{M}^{[\mathbf{N}]}$ that corresponds to $M_N \triangleq [m_{ij}]$ is defined by

$$\mathbf{M}^{[\mathbf{N}]}(\vartheta,\varphi) = \sum_{i=1}^N\sum_{j=1}^N \mathbb{1}_{P_i}(\vartheta)\mathbb{1}_{P_j}(\varphi)m_{ij}, \quad (\vartheta,\varphi) \in [0,1]^2,$$

where $\mathbb{1}_{P_i}(\cdot)$ is the indicator function (that is, $\mathbb{1}_{P_i}(\vartheta) = 1$ if $\vartheta \in P_i$ and $\mathbb{1}_{P_i}(\vartheta) = 0$ if $\vartheta \notin P_i$). Let $\mathbf{x}^{[\mathbf{N}]}(t) \in (L^2[0,1])^n$ be the piece-wise constant function (in the ϑ argument) corresponding to $x(t) \triangleq (x_1(t)^\intercal, ..., x_N(t)^\intercal)^\intercal \in \mathbb{R}^{nN}$ that is given by $\mathbf{x}_\vartheta^{[\mathbf{N}]}(t) \triangleq \sum_{i=1}^N \mathbb{1}_{P_i}(\vartheta)x_i(t),\ \forall\vartheta \in [0,1]$. Similarly, define $\mathbf{u}^{[\mathbf{N}]}(t) \in (L^2[0,1])^n$ that corresponds to $u(t) \triangleq (u_1(t)^\intercal, ..., u_N(t)^\intercal)^\intercal \in \mathbb{R}^{nN}$.

Then the network system in (8) may be compactly represented by the following graphon dynamical system

$$\begin{aligned}\dot{\mathbf{x}}^{[\mathbf{N}]}(t) = &\Big[A(t)\mathbb{I} + D(t)\mathbf{M}^{[\mathbf{N}]}\Big]\,\mathbf{x}^{[\mathbf{N}]}(t)\\ &+ \Big[B(t)\mathbb{I} + D(t)\mathbf{M}^{[\mathbf{N}]}\Big]\,\mathbf{u}^{[\mathbf{N}]}(t), \quad t \in [0,T],\end{aligned} \tag{9}$$

where $\mathbf{x}^{[\mathbf{N}]}(t), \mathbf{u}^{[\mathbf{N}]}(t) \in (L^2_{pwc[0,1]})^n$, $\mathbf{M}^{[\mathbf{N}]} \in \mathcal{W}_c$ represents the step function graphon associated with the underlying graph adjacency matrix M_N, and $(L^2_{pwc}[0,1])^n$ denotes the set of all piece-wise constant (over each element of the uniform partition) functions in $(L^2[0,1])^n$.

The trajectories of the graphon dynamical system in (9) correspond one-to-one to the trajectories of the network system in (8), following a similar proof argument to [16, Lemma 3]. Moreover, the system in (5) can represent the limit system for a sequence of systems represented in the form of (9) when the underlying step function graphon sequence converges to a limit graphon under the operator norm and the sequence of initial conditions converges to a limit initial condition in $(L^2[0,1])^n$ (following a similar proof argument to [16, Theorem 7]).

IV. LQG Graphon Mean Field Games

A. Stochastic Dynamic Games on Finite Networks

Consider an N-node graph where each node is associated with a homogeneous population of agents. Each individual agent is influenced by the mean field of its nodal population and the mean fields of other nodal populations over the graph. Let $\mathcal{V}_c$ denote the set of nodes and $N = |\mathcal{V}_c|$ denote the total number of nodes. Let $\mathcal{C}_q$ denote the set of agents in the qth cluster. Then the total number of agents is $K = \sum_{q=1}^N |\mathcal{C}_q|$.

Following the problem formulation in [1]–[3], the dynamics of an individual agent $i \in \{1,...,K\}$ are given by

$$dx_i(t) = (Ax_i(t) + Bu_i(t) + Dz_i(t))dt + \Sigma dw_i(t), \tag{10}$$

where $t \in [0,T]$, $x_i(t)$, $u_i(t)$, and $z_i(t)$ are respectively the state, the control and the network influence in $\mathbb{R}^n$. $\{w_i, 1 \le i \le K\}$ are independent standard n-dimensional Wiener processes and are independent of the initial conditions $\{x_i(0), 1 \le i \le K\}$ which are also assumed to be independent. Σ is a constant $n\times n$ matrix. We drop the time index for $A(\cdot), B(\cdot), D(\cdot)$ purely for notation simplicity. For an agent $i \in \mathcal{C}_q$, the network influence is given by

$$z_i(t) = \frac{1}{N}\sum_{\ell=1}^N m_{q\ell}\frac{1}{|\mathcal{C}_\ell|}\sum_{j\in\mathcal{C}_\ell} x_j(t) = \frac{1}{N}\sum_{\ell=1}^N m_{q\ell}\bar{x}_\ell(t) \tag{11}$$

where $M_N = [m_{q\ell}]$ is the adjacency matrix of the underlying graph, $\bar{x}_\ell(t) \triangleq \int_{\mathbb{R}^n} x\hat{\mu}_\ell(t,x)dx$ with $\hat{\mu}_\ell(t,\cdot)$ as the empirical distribution of agent states in cluster $\mathcal{C}_\ell$ at time t. The

[1] That is, the solution $\mathbf{x}$ is continuous on $[0,T]$, $\mathbf{x}(t)$ lies in the domain of $[A(t)\mathbb{I} + D(t)\mathbf{M}]$ for all $t \in [0,T]$, $\mathbf{x}$ is continuously differentiable on $(0,T]$ and satisfies (5).

individual cost for agent i is given by

$$J_i(u_i) \triangleq \mathbb{E}\Big(\int_0^T \big(\|x_i(t) - \nu_i(t)\|_Q^2 + \|u_i(t)\|_R^2\big) dt + \|x_i(T) - \nu_i(T)\|_{Q_T}^2 \Big) \quad (12)$$

where $Q, Q_T \geq 0, R > 0$, $\nu_i(t) \triangleq H(z_i(t) + \eta)$, $\eta \in \mathbb{R}^n$ and $H \in \mathbb{R}^{n\times n}$.

Let $\gamma_i(\cdot,\cdot) : [0,T] \times \mathcal{I}_i \to \mathbb{R}^n$ denote the strategy of agent i, $i \in \{1,...,K\}$ where $\mathcal{I}_i$ denotes the information set available to agent i. The control action of agent i at time t is then given by $u_i(t) = \gamma_i(t,\eta)$ with $\eta \in \mathcal{I}_i$. A strategy K-tuple $(\gamma_1,...,\gamma_K)$ is a *Nash equilibrium* if it satisfies that for all $i \in \{1,\dots,K\}$,

$$J_i(\gamma_i, \gamma_{-i}) \leq J_i(\gamma, \gamma_{-i}), \quad \forall \gamma(\cdot,\cdot) : [0,T] \times \mathcal{I}_i \to \mathbb{R}^n,$$

where $\gamma_{-i} \triangleq (\gamma_1,...,\gamma_{i-1},\gamma_{i+1},...,\gamma_K)$, and $J(\gamma,\gamma_{-i})$ denotes the cost for agent i when agent i follows strategy $\gamma(\cdot,\cdot) : [0,T] \times I_i \to \mathbb{R}^n$ and all the other agents follow strategies specified in γ_{-i}. Given that all other agents are taking strategies specified by γ_{-i}, the *best response* of agent i is defined by $\arg\inf_{\gamma\in\mathcal{U}_i} J_i(\gamma,\gamma_{-i})$, where the sets of admissible strategies $(\mathcal{U}_i)_{i=1}^K$ may consist of open-loop, close-loop, or state-feedback strategies depending on the information structures (see [19] for detailed discussions).

Directly finding Nash equilibria for such problems on large-scale networks is generally intractable. To tackle this the graphon mean field game approach [1]–[3] employs the idea of finding approximate solutions based on both the mean field limit and the graphon limit. The corresponding best response for each individual agent in the approximate solution is decentralized in the sense that only local state observation is required in $\mathcal{I}_i$.

B. Infinite Nodal Population Problems on Finite Networks

In the asymptotic local population limit (i.e. $|\mathcal{C}_q| \to \infty$ for all $q \in \{1,...,N\}$), the dynamics of a generic agent α in the cluster $\mathcal{C}_q$ (i.e. $\alpha \in \mathcal{C}_q$) in the problem are then given by

$$dx_\alpha(t) = (Ax_\alpha(t) + Bu_\alpha(t) + Dz_\alpha(t))dt + \Sigma dw_\alpha(t) \quad (13)$$

where $z_\alpha(t) = \frac{1}{N}\sum_{\ell=1}^N m_{q\ell}\bar{x}_\ell(t)$ and

$$\bar{x}_\ell(t) \triangleq \lim_{|\mathcal{C}_\ell|\to\infty} \frac{1}{|\mathcal{C}_\ell|} \sum_{j\in\mathcal{C}_\ell} x_j(t) = \int_{\mathbb{R}^n} x\mu_\ell(t,dx).$$

with $\mu_\ell(t,\cdot)$ as the probability measure at cluster $\mathcal{C}_\ell$ at time t. The cost for a generic agent $\alpha \in \mathcal{C}_q$ is then given by

$$J_\alpha(u_\alpha) = \mathbb{E}\Big(\int_0^T \big(\|x_\alpha(t) - \nu_\alpha(t)\|_Q^2 + \|u_\alpha(t)\|_R^2\big) dt + \|x_\alpha(T) - \nu_\alpha(T)\|_{Q_T}^2 \Big) \quad (14)$$

where $Q, Q_T \geq 0, R > 0$ and $\nu_\alpha(t) \triangleq H(z_\alpha(t) + \eta)$.

For cluster $\mathcal{C}_\ell$, let $\bar{z}_\ell(t) \triangleq \lim_{|\mathcal{C}_\ell|\to\infty} \frac{1}{|\mathcal{C}_\ell|}\sum_{j\in\mathcal{C}_\ell} z_j(t)$. Let $\bar{z}(t) \triangleq (\bar{z}_1(t)^\intercal,...,\bar{z}_N(t)^\intercal)^\intercal$, and let $\bar{s}(t)$ and $\bar{x}(t)$ be represented similar to $\bar{z}(t)$. Let I_N denote the identity matrix of dimension $N \times N$.

Proposition 1 ([34]) *If there exists a unique solution pair $(\bar{s},\bar{z})$ to the following coupled forward-backward equations*

$$\begin{aligned}
-\dot{\bar{s}}(t) &= I_N \otimes \big(A - BR^{-1}B^\intercal\Pi_t\big)^\intercal \bar{s}(t) \\
&\quad - I_N \otimes (QH - \Pi_t D)\bar{z}(t) - (I_N \otimes QH)(\mathbf{1}_n \otimes \eta) \\
\bar{s}(T) &= (I_N \otimes Q_T H)(\bar{z}(T) + \mathbf{1}_n \otimes \eta),
\end{aligned} \quad (15)$$

$$\begin{aligned}
\dot{\bar{z}}(t) &= I_N \otimes (A - BR^{-1}B^\intercal\Pi_t)\bar{z}(t) \\
&\quad + \frac{1}{N}M_N \otimes D\bar{z}(t) - \frac{1}{N}M_N \otimes BR^{-1}B^\intercal\bar{s}(t) \\
\bar{z}(0) &= \frac{1}{N}(M_N \otimes I_n)\bar{x}(0),
\end{aligned} \quad (16)$$

where $t \in [0,T]$ and $\Pi_{(\cdot)}$ is given by the $n \times n$-dimensional matrix Riccati equation

$$-\dot{\Pi}_t = A^\intercal\Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\intercal\Pi_t + Q, \ \Pi_T = Q_T, \quad (17)$$

then the game problem defined by (13) *and* (14) *has a unique Nash equilibrium and the best response in the equilibrium is given as follows: for a generic agent α in cluster $\mathcal{C}_q$,*

$$u_\alpha(t) = -R^{-1}B^\intercal(\Pi_t x_\alpha(t) + \bar{s}_q(t)), \qquad \alpha \in \mathcal{C}_q. \quad (18)$$

□

The solution pair to the two joint equations (15) and (16) together with the sufficient conditions for the existence and uniqueness can be provided based on the standard fixed point method (see for instance [27]) or the solution method based on Riccati equations following [35]–[37]. See [34] for two computational algorithms.

Each individual agent, in order to generate the mean field best response in (18), needs to solve two nN-dimensional equations (15) and (16), and moreover each agent is required to know the exact graph structure. To overcome these difficulties, we employ the idea of approximating large graph structures by their graphon limit(s) in the following section.

C. Infinite Nodal Population Problems on Graphons

Consider a uniform partition $\{P_1,\dots,P_N\}$ of $[0,1]$ with $P_1 = [0,\frac{1}{N}]$ and $P_k = (\frac{k-1}{N},\frac{k}{N}]$ for $2 \leq k \leq N$. Let node q be associated with the partition P_q. If we embed the functions $\bar{z}$ and $\bar{s}$ into the Hilbert space $L^2\big([0,T];(L^2[0,1])^n\big)$, denoted by $\mathbf{s}^{[\mathbf{N}]}$ and $\mathbf{z}^{[\mathbf{N}]}$ following Section III-B, then the joint equations (15) and (16) can be equivalently represented by the following graphon time-varying dynamical systems:

$$\begin{aligned}
\dot{\mathbf{s}}^{[\mathbf{N}]}(t) &= -\big[\mathbb{A}(t)^\intercal\big]\mathbf{s}^{[\mathbf{N}]}(t) + [(QH - \Pi_t D)\mathbb{I}]\mathbf{z}^{[\mathbf{N}]}(t) \\
&\quad + [QH\mathbb{I}](\eta\mathbf{1}), \ \mathbf{s}^{[\mathbf{N}]}(T) = [Q_T H\mathbb{I}](\mathbf{z}^{[\mathbf{N}]}(T) + \eta\mathbf{1}),
\end{aligned} \quad (19)$$

$$\begin{aligned}
\dot{\mathbf{z}}^{[\mathbf{N}]}(t) &= [\mathbb{A}(t) + D\mathbf{M}^{[\mathbf{N}]}]\mathbf{z}^{[\mathbf{N}]}(t) - [BR^{-1}B^\intercal\mathbf{M}^{[\mathbf{N}]}]\mathbf{s}^{[\mathbf{N}]}(t) \\
\mathbf{z}^{[\mathbf{N}]}(0) &= \int_{[0,1]} \mathbf{M}^{[\mathbf{N}]}(\cdot,\beta)\bar{x}_\beta(0)d\beta \in (L^2[0,1])^n
\end{aligned} \quad (20)$$

where $\mathbb{A}(t) \triangleq [(A - BR^{-1}B^\intercal\Pi_t)\mathbb{I}]$, and $\mathbf{s}^{[\mathbf{N}]}, \mathbf{z}^{[\mathbf{N}]} \in L^2([0,T];(L^2_{pwc}[0,1])^n)$.

This equivalent formulation enables us to represent the joint equations (15) and (16) on arbitrary-size graphs, since any graph of a finite size can be represented by $\mathbf{M}^{[\mathbf{N}]}$ through a step function graphon as illustrated in Section III-B.

As the number of nodes goes to infinity, the limit of joint equations (15) and (16) (if existing) is given by the following joint equations (21) and (22). (The existence, uniqueness and convergence properties of the solutions are presented later in detail in Theorem 1).

The Global LQG-GMFG Forward-Backward Equations

$$\begin{aligned}&\dot{\mathbf{s}}(t) = -\big[\mathbb{A}(t)^{\intercal}\big]\mathbf{s}(t) + [(QH - \Pi_t D)\mathbb{I}]\mathbf{z}(t) + [QH\mathbb{I}](\eta\mathbf{1}),\\ &\mathbf{s}(T) = [Q_T H\mathbb{I}](\mathbf{z}(T) + \eta\mathbf{1}) \in (L^2[0,1])^n,\end{aligned} \tag{21}$$

$$\begin{aligned}&\dot{\mathbf{z}}(t) = [\mathbb{A}(t) + D\mathbf{M}]\mathbf{z}(t) - [BR^{-1}B^{\intercal}\mathbf{M}]\mathbf{s}(t),\\ &\mathbf{z}(0) = \int_{[0,1]}\mathbf{M}(\cdot,\beta)\bar{x}_\beta(0)d\beta \in (L^2[0,1])^n,\end{aligned} \tag{22}$$

where $\mathbb{A}(t) \triangleq [(A - BR^{-1}B^{\intercal}\Pi_t)\mathbb{I}]$, $\Pi_{(\cdot)}$ is given by

$$-\dot{\Pi}_t = A^{\intercal}\Pi_t + \Pi_t A - \Pi_t BR^{-1}B^{\intercal}\Pi_t + Q,\ \Pi_T = Q_T, \tag{23}$$

$\mathbf{s}(t), \mathbf{z}(t) \in (L^2[0,1])^n$ for all $t \in [0,T]$, and $\mathbf{s},\mathbf{z} \in L^2([0,T];(L^2[0,1])^n)$.

If the joint solutions $\mathbf{s}$ and $\mathbf{z}$ to (21) and (22) exist in $L^2([0,T];(L^2[0,1])^n)$, then by Lemma 1 they also lie in $C\big([0,T];(L^2[0,1])^n\big)$. By the Arzelà–Ascoli Theorem and the Uniform Limit Theorem [38], the space $C([0,T];(L^2[0,1])^n)$ is complete under the uniform norm $\|\cdot\|_C$ defined by

$$\|\mathbf{v}\|_{C} \triangleq \sup_{t\in[0,T]}\|\mathbf{v}(t)\|_{(L^2[0,1])^n},\ \forall\mathbf{v} \in C([0,T];(L^2[0,1])^n). \tag{24}$$

Proposition 2 ([34]) *Assume there exists a unique classical solution pair* $(\mathbf{s},\mathbf{z})$ *to equations* (21) *and* (22). *Then the graphon limit game problem has a unique Nash equilibrium and the best response in the equilibrium for a generic agent* α *in cluster* $\mathcal{C}_\vartheta$ *for almost all* $\vartheta \in [0,1]$ *is given by*

$$u_\alpha(t) = -R^{-1}B^{\intercal}(\Pi_t x_\alpha(t) + \mathbf{s}_\vartheta(t)),\ \alpha \in \mathcal{C}_\vartheta, \vartheta \in [0,1] \tag{25}$$

where $(\mathbf{s}_\vartheta(t))_{\vartheta\in[0,1],t\in[0,T]}$ *is given by the joint equations* (21) *and* (22), *and* Π *is given by* (23). □

The best response in the Nash equilibrium for the limit problem is the same as that in [1]–[3], but the characterization of the offset process $\mathbf{s}$ is different. The Global LQG-GMFG Forward-Backward Equations explicitly specify the space for the solution pair $(\mathbf{s},\mathbf{z})$ following similar lines of analysis in Graphon Control in [14]–[16], whereas in [1]–[3] these processes are specified in a point-wise manner without specifying the space for the global processes $\mathbf{z}$ and $\mathbf{s}$. The formulation in this paper further enables the analysis of LQG-GMFG solutions based on subspace decompositions and the convergence analysis of finite network mean fields to the limit graphon mean field later in Theorem 1.

V. Solutions Based on the Fixed-Point Analysis

A. Existence, Uniqueness and Convergence

Let $\mathbb{A}(t) \triangleq [(A - BR^{-1}B^{\intercal}\Pi_t)\mathbb{I}]$. Let $\phi_1^{\mathbf{M}}(\cdot,\cdot)$ and $\phi_2(\cdot,\cdot)$ denote the evolution operators ([33, Chapter 5]) respectively relative to $[\mathbb{A}(\cdot) + D\mathbf{M}]$ and $[-\mathbb{A}(\cdot)^{\intercal}]$. Following [33, Theorem 5.2, Chapter 5], the evolution operators satisfy

$$\begin{aligned}\frac{d\phi_1^{\mathbf{M}}(t,\tau)}{dt} &= [\mathbb{A}(t) + D\mathbf{M}]\phi_1^{\mathbf{M}}(t,\tau),\ \phi_1^{\mathbf{M}}(\tau,\tau) = \mathbb{I},\\ \frac{d\phi_2(t,\tau)}{dt} &= [-\mathbb{A}(t)^{\intercal}]\phi_2(t,\tau),\quad \phi_2(\tau,\tau) = \mathbb{I},\end{aligned}$$

in $\mathcal{L}_u\big((L^2[0,1])^n\big)$ (the space of all bounded linear operators on $(L^2[0,1])^n$ under the uniform operator topology). Define the mapping $\mathrm{L}_0(\cdot) : \mathcal{W}_c \to [0,\infty)$ as follows:

$$\begin{aligned}\mathrm{L}_0(\mathbf{M}) \triangleq \sup_{t\in[0,T]}\bigg\{\int_0^t\int_\tau^T \Big\|\big\{\phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^{\intercal}\mathbf{M}]\\ \phi_2(\tau,q)[(QH - \Pi_q D)\mathbb{I}]\big\}\Big\|_{\text{op}}dqd\tau\bigg\} +\\ \sup_{t\in[0,T]}\bigg\{\int_0^t\Big\|\phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^{\intercal}\mathbf{M}]\phi_2(\tau,T)[Q_T H\mathbb{I}]\Big\|_{\text{op}}d\tau\bigg\}\end{aligned} \tag{26}$$

for any $\mathbf{M} \in \mathcal{W}_c$. With a slight abuse of notation, we use $\|\cdot\|_{\text{op}}$ to denote the operator norm for both $\mathcal{L}((L^2[0,1])^n)$ and $\mathcal{L}(L^2[0,1])$, as it will become clear in the specific context which operator norm is referred to.

Theorem 1 ([34] Network MF to Limit Graphon MF) *If there exists a constant* c_0 *(*$0 \le c_0 < 1$*) such that*

$$\mathrm{L}_0(\mathbf{M}) \le c_0 \quad \textit{and} \quad \mathrm{L}_0(\mathbf{M}^{[\mathbf{N}]}) \le c_0 \textit{ for all } N, \tag{27}$$

$$\textit{and} \lim_{N\to\infty}\|\mathbf{M} - \mathbf{M}^{[\mathbf{N}]}\|_{\text{op}} = 0,\quad \lim_{N\to\infty}\|\mathbf{z}(0) - \mathbf{z}^{[\mathbf{N}]}(0)\|_2 = 0, \tag{28}$$

then (i) there exist a unique classical solution pair $(\mathbf{s}^{[\mathbf{N}]},\mathbf{z}^{[\mathbf{N}]})$ *to the joint equations* (19) *and* (20) *for each* N *and a unique classical solution pair* $(\mathbf{s},\mathbf{z})$ *to the joint equations* (21) *and* (22) *in the product space* $C([0,T];(L^2[0,1])^n) \times C([0,T];(L^2[0,1])^n)$ *under the uniform norm; (ii)*

$$\lim_{N\to\infty}\|\mathbf{s} - \mathbf{s}^{[\mathbf{N}]}\|_C = 0 \ \textit{ and } \ \lim_{N\to\infty}\|\mathbf{z} - \mathbf{z}^{[\mathbf{N}]}\|_C = 0; \tag{29}$$

(iii) moreover, the asymptotic errors are given by

$$\begin{aligned}&\|\mathbf{z} - \mathbf{z}^{[\mathbf{N}]}\|_{C}\\ &\qquad = O\Big\{\max(\|\mathbf{M} - \mathbf{M}^{[\mathbf{N}]}\|_{\text{op}},\ \|\mathbf{z}(0) - \mathbf{z}^{[\mathbf{N}]}(0)\|_2)\Big\},\\ &\|\mathbf{s} - \mathbf{s}^{[\mathbf{N}]}\|_{C}\\ &\qquad = O\Big\{\max(\|\mathbf{M} - \mathbf{M}^{[\mathbf{N}]}\|_{\text{op}},\ \|\mathbf{z}(0) - \mathbf{z}^{[\mathbf{N}]}(0)\|_2)\Big\}.\end{aligned}$$

□

Remark 2 In the characterization of the graphon convergence in (28), the cut norm $\|\cdot\|_\square$ may also be employed, since for any $\mathbf{M} \in \mathcal{W}_1$ the following inequalities hold

$$\frac{1}{8}\|\mathbf{M}\|_{\text{op}}^2 \le \|\mathbf{M}\|_\square \le \|\mathbf{M}\|_{\text{op}} \tag{30}$$

where $\|\mathbf{M}\|_\square \triangleq \sup_{S,T\subset[0,1]}\Big|\int_{S\times T}\mathbf{M}(x,y)dxdy\Big|$. The two inequalities in (30) are immediate consequences of [39, Lemma E.6 and Eq. (4.4)]. The *cut metric* $\delta_\square(\cdot,\cdot)$ is defined by $\delta_\square(\mathbf{U},\mathbf{V}) = \inf_{\phi\in\Phi}\|\mathbf{U}^\phi - \mathbf{V}\|_\square$ for any $\mathbf{U},\mathbf{V} \in \mathcal{W}_c$, where Φ denotes the set of all measure preserving transformations from $[0,1]$ to $[0,1]$ and $\mathbf{U}^{\phi}(x,y) \triangleq \mathbf{U}(\phi(x),\phi(y))$ (see [6]). Following the procedures in [6, p.157], random graphs are generated as follows: first sample N points $\{x_1,...,x_N\}$ randomly from the uniform distribution in $[0,1]$ and then connect all the unordered nodes pairs (i,j), $i\neq j$, with probability $\mathbf{M}(x_i,x_j)$ (or with weight $\mathbf{M}(x_i,x_n)$). If the finite graphs (with the associated step function graphons $\{\mathbf{M}^{[\mathbf{N}]}\}$) are random graphs generated from an underlying graphon $\mathbf{M}\in\mathcal{W}_0\subset\mathcal{W}_1$ following the procedures above, the upper bound of the asymptotic error to the graphon limit in the cut metric given by [6, Lemma 10.16] is $O(\frac{1}{\sqrt{\log N}})$ with probability at least $1-\exp(-\frac{N}{2\log N})$ under the optimal $\phi^*\in\Phi$ required by the cut metric. Following Theorem 1 and (30), under the conditions in (27), the asymptotic error for $\|\mathbf{z}-\mathbf{z}^{[\mathbf{N}]}\|_{C}$ and $\|\mathbf{s}-\mathbf{s}^{[\mathbf{N}]}\|_{C}$ in this case is then

$$O\left\{\max\left((\log N)^{-\frac{1}{4}},\ \|\mathbf{z}(0)-\mathbf{z}^{[\mathbf{N}]}(0)\|_2\right)\right\} \tag{31}$$

with probability at least $1-\exp(-\frac{N}{2\log N})$). The probability here is due to the randomness in the sampling procedure to generate random graphs in [6, p.157]. □

B. Spectral Decompositions of the Joint Equations

Consider all the normalized eigenfunctions $\{\mathbf{f}_\ell\}_{\ell\in\mathcal{I}_\lambda}$ of $\mathbf{M}$ associated with eigenvalues $\{\lambda_\ell\}_{\ell\in\mathcal{I}_\lambda}$, where $\mathcal{I}_\lambda$ denotes the index set for all the non-zero eigenvalues (allowing repeated eigenvalues) of $\mathbf{M}$. Since the graphon operator $\mathbf{M}$ defined as (1) is a Hilbert–Schmidt integral operator and hence a compact operator in $\mathcal{L}(L^2[0,1])$, the number of elements in $\mathcal{I}_\lambda$ can be finite or countably infinite (see for instance [12, Proposition 1]). Let $\mathcal{S}=\operatorname{span}(\mathbf{f}_\ell,\ell\in\mathcal{I}_\lambda)$ and let $\mathcal{S}^\perp$ denote the orthogonal complement of $\mathcal{S}$ in $L^2[0,1]$. Projecting the processes $\mathbf{z}$ and $\mathbf{s}$ governed by (21) and (22) into the orthogonal subspaces $(\mathcal{S}^\perp)^n$ and $(\operatorname{span}(\mathbf{f}_\ell))^n\subset(\mathcal{S})^n$ for all $\ell\in\mathcal{I}_\lambda$ yields the following result.

Proposition 3 ([34]) *If the Global LQG-GMFG Forward-Backward Equations* (21) *and* (22) *have a unique classical solution pair* $(\mathbf{s},\mathbf{z})$*, then the solution pair satisfies the following: for almost all $\theta\in[0,1]$ and for all $t\in[0,T]$,*

$$\begin{aligned}\mathbf{s}_\theta(t)&=\sum_{\ell\in\mathcal{I}_\lambda}\mathbf{f}_\ell(\theta)s^\ell(t)+\breve{s}(t)(1-\sum_{\ell\in\mathcal{I}_\lambda}\mathbf{f}_\ell^\intercal\mathbf{1}\mathbf{f}_\ell(\theta))),\\ \mathbf{z}_\theta(t)&=\sum_{\ell\in\mathcal{I}_\lambda}\mathbf{f}_\ell(\theta)z^\ell(t),\end{aligned} \tag{32}$$

where for all $\ell\in\mathcal{I}_\lambda$, $s^\ell(t)\mathbf{f}_\ell\in(\operatorname{span}(\mathbf{f}_\ell))^n$ and $z^\ell(t)\mathbf{f}_\ell\in(\operatorname{span}(\mathbf{f}_\ell))^n$, $\breve{s}(t)(\mathbf{1}-\sum_{\ell\in\mathcal{I}_\lambda}\mathbf{f}_\ell^\intercal\mathbf{1}\mathbf{f}_\ell)\in(\mathcal{S}^\perp)^n$, and s^ℓ, z^ℓ and $\breve{s}\in C([0,T];\mathbb{R}^n)$ are given by

$$\begin{aligned}\dot{s}^\ell(t)=&-(A-BR^{-1}B^\intercal\Pi_t)s^\ell(t)+[QH-\Pi_tD]z^\ell(t)\\ &+QH\eta,\quad s^\ell(T)=Q_TH(z^\ell(T)+\eta),\quad \ell\in\mathcal{I}_\lambda,\end{aligned} \tag{33}$$

$$\begin{aligned}&\dot{z}^\ell(t)=(A-BR^{-1}B^\intercal\Pi_t+\lambda_\ell D)z^\ell(t)-\lambda_\ell BR^{-1}B^\intercal s^\ell(t),\\ &z^\ell(0)=\lambda_\ell\int_{[0,1]}\mathbf{f}_\ell(\beta)\bar{x}_\beta(0)d\beta,\quad \ell\in\mathcal{I}_\lambda,\end{aligned} \tag{34}$$

$$\dot{\breve{s}}(t)=-(A-BR^{-1}B^\intercal\Pi_t)\breve{s}(t)+QH\eta,\quad \breve{s}(T)=Q_TH\eta. \tag{35}$$

□

Remark 3 (Solution Complexity) It is worth emphasizing that (33), (34) and (35) are all n-dimensional differential equations, that is, $z^\ell(t)$, $s^\ell(t)$ and $\breve{s}(t)$ are n-dimensional vectors. The solution pair to the joint equations (33) and (34) can be numerically computed via fixed-point iterations (see Algorithm 1 in [34]). Each agent only needs to solve d_{dist} number of forward-backward coupled n-dimensional equations as (33) and (34), and one n-dimensional differential equation as (35), where d_{dist} denotes the number of distinct non-zero eigenvalues of $\mathbf{M}$. We note that $d_{\text{dist}}\le\operatorname{rank}(\mathbf{M})$. If d_{dist} is infinite, one may rely on approximations via a finite number of eigendirections. A special case of the joint equations (33) and (34) is studied in [24]. □

VI. Solutions Based on Riccati Equations

A. Decoupling Joint Equations Based on Riccati Equations

Extending the idea for decoupling finite dimensional coupled forward-backward differential equations in [35]–[37] to the infinite dimensional case, we may decouple equations (21) and (22) based on the following non-symmetric operator Riccati equation

$$\begin{aligned}-\dot{\mathbb{P}}=\mathbb{A}(t)^\intercal\mathbb{P}+\mathbb{P}\mathbb{A}(t)+\mathbb{P}[D\mathbf{M}]-\mathbb{P}[BR^{-1}B^\intercal\mathbf{M}]\mathbb{P}\\ -[(QH-\Pi_tD)\mathbb{I}],\quad \mathbb{P}(T)=[Q_TH\mathbb{I}]\end{aligned} \tag{36}$$

where $\mathbb{A}(t)=(A-BR^{-1}B^\intercal\Pi_t)\mathbb{I}$ and $\Pi_{(\cdot)}$ is the solution to the matrix differential Riccati equation in (23). Consequently we formulate the following assumption:

(A1) The operator Riccati equation (36) has a unique mild solution[2] $\mathbb{P}$ in $C_s\big([0,T];\mathcal{L}\big((L^2[0,1])^n\big)$.

Proposition 4 ([34]) *If Assumption* (A1) *holds, then the joint equations* (21) *and* (22) *have a unique classical solution pair.*

Given the solution to (36), the proof in [34] actually provides a direct computation procedure for decoupling and solving the joint equations (21) and (22) by introducing a new process $\mathbf{e}(t)=\mathbf{s}(t)-\mathbb{P}\mathbf{z}(t), t\in[0,T]$ (see [34] for details).

Proposition 5 ([34]) *If* $\mathrm{L}_0(\mathbf{M})<1$*, then* (A1) *holds.* □

B. Subspace Decomposition for Riccati Equations

Let the subspace $\mathcal{S}\subset L^2[0,1]$ be the graphon invariant subspace of $\mathbf{M}$ defined in Section II-B and let $\mathcal{S}^\perp$ denote its orthogonal complement subspace in $L^2[0,1]$. $\bar{\mathbb{T}}\in\mathcal{L}((L^2[0,1])^n)$ is called the $(\mathcal{S})^n$*-equivalent operator* of $\mathbb{T}\in\mathcal{L}((L^2[0,1])^n)$ in the subspace if

$$\bar{\mathbb{T}}\mathbf{v}=\mathbb{T}\mathbf{v}\text{ and }\bar{\mathbb{T}}\mathbf{u}=0,\quad\forall\mathbf{v}\in(\mathcal{S})^n,\ \forall\mathbf{u}\in(\mathcal{S}^\perp)^n. \tag{37}$$

Let $\bar{\mathbb{P}}(t)\in\mathcal{L}((L^2[0,1])^n)$ denote the $(\mathcal{S})^n$-equivalent operator of $\mathbb{P}(t)$. Let $\mathbb{I}_{\mathcal{S}}$ (resp. $\mathbb{I}_{\mathcal{S}^\perp}$) in $\mathcal{L}(L^2[0,1])$ denote the

[2]That is, $\mathbb{P}\in C_s([0,T];\mathcal{L}((L^2[0,1])^n)$ and satisfies the following equation for all $\mathbf{v}\in(L^2[0,1])^n$, $\mathbb{P}(t)\mathbf{v}=\mathbb{P}(T)\mathbf{v}+\int_t^T\Big(\mathbb{A}(\tau)^\intercal\mathbb{P}(\tau)+\mathbb{P}(\tau)(\mathbb{A}(\tau)+[D\mathbf{M}])-\mathbb{P}(\tau)[BR^{-1}B^\intercal\mathbf{M}]\mathbb{P}(\tau)-[(QH-\Pi_\tau D)\mathbb{I}]\Big)\mathbf{v}d\tau$ with terminal condition $P(T)=[Q_TH\mathbb{I}]$.

$\mathcal{S}$-equivalent operator (resp. $\mathcal{S}^\perp$-equivalent operator) of the identity operator $\mathbb{I} \in \mathcal{L}(L^2[0,1])$.

Theorem 2 ([34] Riccati Eqn. Subspace Decomposition) *If* (A1) *holds, then the solution to the non-symmetric operator Riccati equation* (36) *is equivalently given by*

$$\mathbb{P}(t) = [P^\perp(t)\mathbb{I}_{\mathcal{S}^\perp}] + \bar{\mathbb{P}}(t) \ \in \mathcal{L}((L^2[0,1])^n), \quad t \in [0,T],$$

where $[P^\perp(t)\mathbb{I}_{\mathcal{S}^\perp}] \in \mathcal{L}((\mathcal{S}^\perp)^n)$, $\bar{\mathbb{P}}(t) \in \mathcal{L}((\mathcal{S})^n)$, $P^\perp(t) \in \mathbb{R}^{n\times n}$ *is given by the matrix differential equation*

$$\begin{aligned} -\dot{P^\perp} &= (A - BR^{-1}B^\intercal \Pi_t)^\intercal P^\perp + P^\perp(A - BR^{-1}B^\intercal \Pi_t) \\ &\quad -(QH - \Pi_t D), \qquad P^\perp(T) = \gamma Q_T, \ t \in [0,T], \end{aligned} \tag{38}$$

and $\bar{\mathbb{P}}(t) \in \mathcal{L}((\mathcal{S})^n)$ *is the mild solution to the non-symmetric operator Riccati equation*

$$\begin{aligned} -\dot{\bar{\mathbb{P}}} &= [(A - BR^{-1}B^\intercal \Pi_t)\mathbb{I}_{\mathcal{S}}]^\intercal \bar{\mathbb{P}} + \bar{\mathbb{P}}[(A - BR^{-1}B^\intercal \Pi_t)\mathbb{I}_{\mathcal{S}}] \\ &\quad + \bar{\mathbb{P}}[D\mathbf{M}] - \bar{\mathbb{P}}[BR^{-1}B^\intercal \mathbf{M}]\bar{\mathbb{P}} - [(QH - \Pi_t D)\mathbb{I}_{\mathcal{S}}], \\ \bar{\mathbb{P}}(T) &= [Q_T H \mathbb{I}_{\mathcal{S}}], \quad t \in [0,T]. \end{aligned}$$

□

Remark 4 The decomposition in Theorem 2 is due to the property that the parameter operators $[(A - BR^{-1}B^\intercal \Pi_t)\mathbb{I}]$, $[D\mathbf{M}]$, $[BR^{-1}B^\intercal \mathbf{M}]$, $[(QH - \Pi_t D)\mathbb{I}]$ and $[Q_T H \mathbb{I}]$ in the Riccati equation (36) share the same orthogonal invariant subspaces $(\mathcal{S})^n$ and $(\mathcal{S}^\perp)^n$ (see [17, Proposition 3]). Such decompositions can be generalized to Riccati equations with general parameter operators in $\mathcal{L}((L^2[0,1])^n)$ where the parameter operators are only required to share some common orthogonal invariant subspaces $(\mathcal{S})^n$ and $(\mathcal{S}^\perp)^n$. □

Corollary 1 ([34] Riccati Eqn. Spectral Decomposition) *Assume* (A1) *holds. Let* $\{\mathbf{f}_\ell\}_{\ell \in \mathcal{I}_\lambda}$ *be the orthonormal eigenfunctions of* $\mathbf{M}$ *where* $\mathcal{I}_\lambda$ *denotes the index set for all the non-zero eigenvalues (allowing repeated eigenvalues) of* $\mathbf{M}$. *Then the solution to the non-symmetric operator Riccati equation* (36) *is equivalently given by*

$$\mathbb{P}(t) = [P^\perp(t)\mathbb{I}] + \sum_{\ell \in \mathcal{I}_\lambda} [(\bar{P}^\ell(t) - P^\perp(t))\mathbf{f}_\ell \mathbf{f}_\ell^\intercal], \quad t \in [0,T],$$

where $\mathbb{P}(t) \in \mathcal{L}((L^2[0,1])^n)$, $[P^\perp(t)\mathbb{I}] \in \mathcal{L}((L^2[0,1])^n)$, $[(\bar{P}^\ell(t) - P^\perp(t))\mathbf{f}_\ell \mathbf{f}_\ell^\intercal] \in \mathcal{L}(\mathcal{S}^n)$, $P^\perp(t) \in \mathbb{R}^{n\times n}$ *and* $\bar{P}^\ell(t) \in \mathbb{R}^{n\times n}$ *are respectively given by* $n \times n$*-dimensional matrix differential equation* (38) *and the following* $n \times n$*-dimensional non-symmetric matrix Riccati equation*

$$\begin{aligned} -\dot{\bar{P}}^\ell &= (A - BR^{-1}B^\intercal \Pi_t)^\intercal \bar{P}^\ell + \bar{P}^\ell(A - BR^{-1}B^\intercal \Pi_t) \\ &\quad + \lambda_\ell \bar{P}^\ell D - \lambda_\ell \bar{P}^\ell BR^{-1}B^\intercal \bar{P}^\ell - (QH - \Pi_t D), \\ \bar{P}^\ell(T) &= Q_T H, \quad \ell \in \mathcal{I}_\lambda, \end{aligned}$$

with λ_ℓ *as the eigenvalue of* $\mathbf{M}$ *corresponding to* $\mathbf{f}_\ell$. □

VII. Illustrative Examples on Random Graphs

Consider the following stochastic block model (SBM) that generates random simple graphs of arbitrary sizes (see for instance [40]) by connecting nodes from three node communities with probabilities specified below:

$$[w_{ij}] = \begin{bmatrix} 0.25 & 0.5 & 0.2 \\ 0.5 & 0.35 & 0.7 \\ 0.2 & 0.7 & 0.4 \end{bmatrix}. \tag{39}$$

For random graph sequences of increasing sizes generated from the SBM above, the associated graphon limit is $\mathbf{M}(x,y) = \sum_{i=1}^3 \sum_{j=1}^3 w_{ij} \mathbb{1}_{P_i}(x)\mathbb{1}_{P_j}(y)$, $(x,y) \in [0,1]^2$, where $\{P_1, P_2, P_3\}$ is a partition of $[0,1]$ and $|P_i|$ is proportional to the size of ith node community. Clearly $\mathrm{rank}(\mathbf{M}) = \mathrm{rank}([w_{ij}]) = 3$.

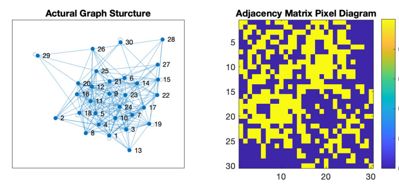

Fig. 2: A 30-node random graph instance generated from SBM in (39), and the associated pixel representation.

The parameters in the simulation are:

$$\begin{aligned} &A = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix}, \ Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \\ &B = D = R = Q_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \eta = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ &T = 1, \ n = 2, \ N = 30, \ |C_\ell| = 4, \ 1 \le \ell \le N. \end{aligned} \tag{40}$$

The initial conditions are independently drawn from Gaussian distributions with variance 1 and mean values that are generated randomly from $[-3,3]$. These mean values are used in computing the graphon mean field game solutions.

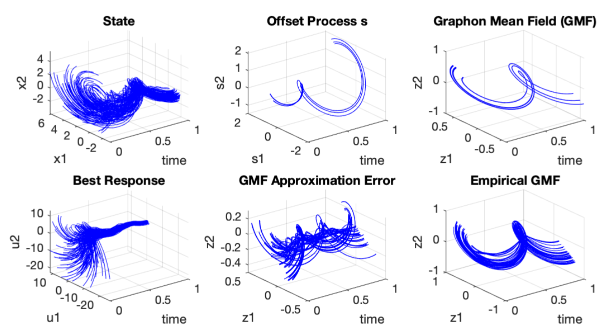

Fig. 3: Simulation on a network generated from the stochastic block model.

The simulation result on the 30-node graph in Fig. 2 is illustrated in Fig. 3. The relative error $\frac{\|\mathbf{z}_E - \mathbf{z}\|_C}{\|\mathbf{z}_E\|_C}$ of the graphon mean field approximation is 29.256% where $\mathbf{z}_E$ is the actual network mean field and $\mathbf{z}$ is the graphon mean field. The error between the graphon limit $\mathbf{M}$ and the step function graphon $\mathbf{M}^{[\mathbf{N}]}$ (associated with the graph) is $\|\mathbf{M} - \mathbf{M}^{[\mathbf{N}]}\|_{\mathrm{op}} = 0.178$ and the graphon limit operator norm is $\|\mathbf{M}\| = 0.434$. The relative approximation error $\frac{\|\mathbf{z}_E - \mathbf{z}\|_C}{\|\mathbf{z}_E\|_C}$ decreases as the size of the graph increases, as illustrated in the results on graphs of different sizes shown in Fig. 4.

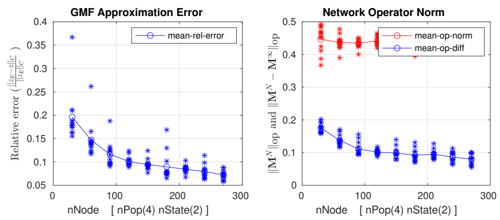

Fig. 4: Graphon mean field game approximation errors on networks of different size. 12 independent simulations are carried out for each size. In the figure on the right, red dots represent values for $\|\mathbf{M}^{[\mathbf{N}]} - \mathbf{M}\|_{\text{op}}$ in different experiments.

VIII. Conclusion

This work studied solution methods for LQG graphon mean field game problems based on subspace and spectral decompositions, and established new asymptotic error bounds on the convergence of the network mean fields to the graphon mean field. Future work should focus on cases with heterogeneous parameters in dynamics, computational procedures for nonlinear graphon mean field games, graphon control theory for nonlinear systems, and the counterpart theory for sparse graphs.

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