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Optimal Network Location in Infinite Horizon LQG Graphon Mean Field Games

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Abstract— **We propose to study a class of infinite horizon linear quadratic Gaussian Graphon Mean Field Games (GMFGs) inspired by the infinite horizon Mean Field Games in [1]. Graphon Mean Field Games (GMFGs) are non-uniform generalizations of Mean Field Games where the non-uniformity of agents is characterized by the nodes on which they are located in a network. Under mild conditions, we obtain for almost every node, an analytical expression for the cost at GMFG equilibrium, and propose a necessary and sufficient condition under which a particular node in the network is associated with the minimal cost at GMFG equilibrium.**

I. INTRODUCTION

This paper adds to the literature on Graphon Mean Field Games which read as Mean Field Games with network-coupled (populations of) agents, see for example [2]–[12]. Graphon Mean Field Games are generalizations of Mean Field Games (see for example [13], [14]), and can be seen as Mean Field Games with agents located on large undirected graphs. This work is an extension of a similar study of infinite horizon GMFGs [15], and both studies draw a lot of inspiration from mean field games with cost localities studied in [16]. The differences between [16] and the current paper are that in [16] each node is assumed to be associated with an individual agent and graphons are not employed. The current paper focuses on establishing explicit analytical results on the cost at equilibrium and characterizing nodes with minimal cost at equilibrium under mild assumptions on the initial conditions and graphon properties.

Graphon Mean Field Games are asymptotic version of finite large population games, with N agents $\mathcal{A}_i, 1 \leq i \leq N < \infty$, which are distributed over a finite network, represented by its adjacency matrix $(g^n_{i,j})_{i,j=1:n}$, with n nodes. We assume that, at each node $l \in \{1, ..., n\}$ of this network, there is a cluster of agents denoted C_l, and the total number of agents is $N = \sum_{l=1}^{n} |C_l|$.

For each agent $\mathcal{A}_i$ in cluster C_k, the coupling term (also called nodal network mean field) governing its interaction via the network with other players, is given by:

$$z_t^{k,n} = \frac{1}{n}\sum_{l=1}^{n} g^n_{k,l}\frac{1}{|C_l|}\sum_{j\in C_l} x_t^j, \quad t \geq 0,\ i \in \{1...N\}.$$

The flow of network mean fields $(z_t^{k,n})_{t\in[0,\infty),k\in\{1...n\}}$ relies on the sectional information $g^n_{k,\bullet}$ of $\mathcal{A}_i$ which represents the view of the network interactions from the position of agents in cluster $C_k,\ k \in \{1, ..., n\}$. From the point of view of any agent $\mathcal{A}_i$ in any cluster C_k, all individuals residing in cluster C_k are symmetric and their average generates an overall impact of that cluster.

Consider the state evolution of the collection of N agents specified by the set of N controlled linear stochastic differential equations (SDEs) over an infinite horizon below. For each agent $\mathcal{A}_i$, its state denoted $x^i(\cdot) \in R$ evolves according to the SDE:

$$dx_t^i = \left(ax_t^i + bu_t^i\right)dt + \sigma dw_t^i, \quad \forall t \geq 0, \tag{1}$$

where $u^i(\cdot) \in R$ denotes the agent's $\mathcal{A}_i$ control input. For simplicity, we assume that the initial state of agent $\mathcal{A}_i$ is $x_0^i \sim \mathcal{N}(m^l, \nu^2)$, whenever $\mathcal{A}_i$ lies in cluster C_l, $l \in \{1, .., n\}$. Assume the real coefficients a, b, m^l with $l \in \{1..., n\},\ \nu > 0,\ \sigma \geq 0$ are known. Let $\{w^i, i = 1, ..., N\}$ be a collection of independent Brownian motions defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions.

We consider a scenario where each agent $\mathcal{A}_i$ in any cluster C_k aims to minimize infinite horizon quadratic costs given by

$$J^N(u^i, u^{-i}) := \mathbb{E}\int_0^\infty e^{-\rho t}\left[r(u_t^i)^2 + \left(x_t^i - z_t^{k,n}\right)^2\right]dt, \tag{2}$$

where $1 \leq i \leq N$, $\rho > 0$, $r > 0$, and u^{-i} denotes the controls of all agents other than $\mathcal{A}_i$. Controls are chosen from the admissible control space defined below,

$$\mathbb{A} := \{u : \Omega \times [0, \infty) \mapsto \mathbb{R} \mid u \text{ is } \mathbb{F}-\text{progressively measurable and } \mathbb{E}\int_0^\infty e^{-\rho t}|u(t)|^2 dt < \infty\}.$$

A solution concept for the population games on networks we defined above is the well-known Nash equilibrium.

Definition 1 (Nash Equilibrium)**.** *A collection of controls, denoted $(u^{i*})_{i=1}^N \in \mathbb{A}^N$, is a Nash equilibrium if and only if any unilateral deviation from $u^{i*} \in \mathbb{A}$ to any other control $u^i \in \mathbb{A}$ does not yield a lower cost, that is,*

$$J_i^N(u^{i*}, u^{-i*}) \leq J_i^N(u^i, u^{-i*}),\ \forall i = 1, ..., N. \tag{3}$$

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Finding a Nash equilibrium in network-coupled population games gets increasingly complex as both the cluster size and the network size grow. In the situation where the network describing the interaction between the agents is uniform (i.e. fully symmetric), the theory of Mean Field Games provides satisfactory answers to this problem (see [17] [18]).

For non-uniform networks, Graphon Mean Field Games model asymptotic limits of population games in the double limit, $n \to \infty$ and $\min_{l=1:n}|C_l| \to \infty$ (observe that it implies that the number of agents, denoted by $N = \sum_{l=1}^{n}|C_l|$, goes to infinity).

We assume that the sequence of networks, represented by adjacency matrices, $(g^n_{i,j})_{i,j=1:n}$, converges, in the cut metric (see [19]), to a unique limit graphon[1] denoted

$$\begin{aligned} g : [0,1]\times[0,1] &\to [0,1] \\ (\alpha,\beta) &\mapsto g(\alpha,\beta). \end{aligned}$$

Graphons are bounded symmetric Lebesgue measurable functions $g : [0,1]\times[0,1] \to [0,1]$ which can be interpreted as weighted graphs on the set of nodes $[0,1]$ (see [19]).

With the network interaction within a cluster being uniform, we deduce that in the infinite cluster size, at all graphon node $\alpha \in [0,1]$, there exists a representative (or typical) agent, denoted $\mathcal{A}_\alpha$ whose state's evolution is given by the SDE: $t \in [0,\infty)$

$$dx^\alpha_t = (ax^\alpha_t + bu^\alpha_t)\,dt + \sigma dw^\alpha_t, \quad x^\alpha_0 \sim \mathcal{N}(m^\alpha,\nu^2). \tag{4}$$

A representative agent $\mathcal{A}_\alpha$ at node α aims at minimizing an infinite horizon quadratic cost given by

$$J(u^\alpha, z^\alpha) := \mathbb{E}\int_0^\infty e^{-\rho t}\big[r(u^\alpha_t)^2 + \big(x^\alpha_t - z^\alpha_t\big)^2\big]dt, \tag{5}$$

where $r,\rho > 0$ and, the nodal graphon mean field denoted by z^α_t , is given by,

$$z^\alpha_t := \int_0^1 g(\alpha,\beta)\mathbb{E}[x^\beta_t]d\beta,\ \forall t\in[0,\infty),\ \forall\alpha\in[0,1]. \tag{6}$$

II. Infinite Horizon LQG-GMFGs

The Linear Quadratic Gaussian Graphon Mean Field Games (LQG-GMFGs) problem:

1) (Mean Field Inputs) Fix a two-parameter deterministic flow of graphon mean fields $\{z^\alpha_t,\ t\in[0,\infty),\ \alpha\in[0,1]\}$.
2) (Control Problems) Find optimal controls, denoted by $u^{\alpha,o} := (u^{\alpha,o}_t)_{t\in[0,\infty)} \in \mathbb{A}$, such that
$$\begin{aligned} J(u^{\alpha,o}, z^\alpha) &= \min_{u^\alpha\in\mathbb{A}} J(u^\alpha, z^\alpha) \qquad (7) \\ &= \min_{u^\alpha\in\mathbb{A}} \mathbb{E}\int_0^\infty e^{-\rho t}\big[r\big(u^\alpha_t\big)^2 + \big(x^\alpha_t - z^\alpha_t\big)^2\big]dt \end{aligned}$$
subject to the following dynamics
$$dx^\alpha_t = (ax^\alpha_t + bu^\alpha_t)\,dt + \sigma dw^\alpha_t,\ x^\alpha_0 \sim \mathcal{N}(m^\alpha,\nu^2). \tag{8}$$
for all $t\in[0,\infty)$ and all $\alpha\in[0,1]$.
3) (Consistency Conditions) Show that the optimal state trajectories $\{x^{\alpha,o}_t,\ t\in[0,\infty),\ \forall\alpha\in[0,1]\}$, satisfy the consistency conditions, for all $(\alpha,t)\in[0,1]\times[0,\infty)$;
$$z^\alpha_t = \int_0^1 g(\alpha,\beta)\mathbb{E}[x^{\beta,o}_t]d\beta. \tag{9}$$

[1]Strictly speaking, the cut metric is a pseudometric and the unique equivalent classes of graphons are defined up to all measure preserving bijections from $[0,1]$ to $[0,1]$ (see [19] for details).

The control problems can be solved following the standard approach described in [1]. Consider the following algebraic Riccati equation:

$$\rho\pi = 2a\pi - \frac{b^2}{r}\pi^2 + 1, \quad r>0,\ \rho>0. \tag{10}$$

The Riccati equation has a unique positive solution

$$\pi = \sqrt{\frac{r^2(\rho-2a)^2}{4b^4} + \frac{r}{b^2}} - \frac{(\rho-2a)\,r}{2b^2} > 0. \tag{11}$$

Consider $C_b([0,\infty))$ the set of bounded continuous functions over the domain $[0,\infty)$. This space endowed with the supremum norm, $|x|_\infty := \sup_{t\in[0,\infty)}|x(t)|$ is a Banach space. Consider $L^2([0,1])$ the set of square integrable functions on the domain $[0,1]$. This space is a Hilbert space, when endowed with the inner product $\langle x,y\rangle = \int_0^1 x(\beta)y(\beta)d\beta$.

Proposition 1. *Assume that there exists a process* $\{s^\alpha_t,\ \alpha\in[0,1],\ t\in[0,\infty)\} \in C_b([0,\infty))\times L^2([0,1])$ *determined by the offset ODE below;*

$$\frac{ds^\alpha_t}{dt} = \Big(-a + \frac{b^2}{r}\pi + \rho\Big)s^\alpha_t + z^\alpha_t. \tag{12}$$

Then, there exist optimal control processes for the infinite horizon optimal control problems above, namely, for all $\alpha\in[0,1]$,

$$u^{\alpha,o}_t = -\frac{b}{r}\big(\pi x^{\alpha,o}_t + s^\alpha_t\big),\ \forall t\in[0,\infty), \tag{13}$$

where the optimal state processes $(x^{\alpha,o}_t)_{t\in[0,T]}$ *are given by the SDEs,*

$$\begin{aligned} dx^{\alpha,o}_t &= \Big[\Big(a - \frac{b^2}{r}\pi\Big)x^{\alpha,o}_t - \frac{b^2}{r}s^\alpha_t\Big]dt + \sigma dw^\alpha_t, \\ x^{\alpha,o}_0 &\sim \mathcal{N}(m^\alpha,\nu^2). \end{aligned}$$

Proof. The proof is a standard application of LQG tracking control theory (see e.g. [1]). □

Proposition 2. *Assume that there exists a process* $\{q^\alpha_t,\ \alpha\in[0,1],\ t\in[0,\infty)\} \in C_b([0,\infty))\times L^2([0,1])$ *determined by the ODE*

$$\frac{dq^\alpha_t}{dt} = -\sigma^2\pi + \frac{b^2}{r}(s^\alpha_t)^2 + \rho q^\alpha_t - (z^\alpha_t)^2. \tag{14}$$

Then, the optimal costs are given, for all $\alpha\in[0,1]$, *by*

$$\begin{aligned} J(u^\alpha, z) &= \pi\mathbb{E}[(x^{\alpha,o}_0)^2] + 2s^\alpha_0\mathbb{E}[x^{\alpha,o}_0] + q^\alpha_0 \\ &= \pi(\nu^2 + (m^\alpha)^2) + 2s^\alpha_0 m^\alpha + q^\alpha_0. \end{aligned} \tag{15}$$

Proof. The proof is also standard for LQG tracking problems. See for example [1]. □

Once the control problems have been solved and their solutions characterized by the two propositions above, we proceed to verify the consistency condition.

Proposition 3. *Let the assumptions of Proposition 1 be in force. The consistency condition (9) is satisfied if and only if, there exists a process $\{z_t^\alpha,\ \alpha \in [0,1],\ t \in [0,\infty)\} \in C_b([0,\infty)) \times L^2([0,1])$ determined by the ODE:*

$$dz_t^\alpha = \left[\left(a - \frac{b^2}{r}\pi\right) z_t^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_t^\beta d\beta\right] dt, \quad (16)$$
$$z_0^\alpha = \int_0^1 g(\alpha,\beta) m^\beta d\beta.$$

Proof. The consistency condition (9) is in fact a fixed point condition on the optimal states. From Proposition 1, we have an SDE representation for these optimal states. Due to the linearity of the problem, the existence of the fixed point is characterized in terms of the existence of solutions to ODEs (16). □

Compiling the three previous propositions, we obtain that the infinite horizon LQG-GMFGs under study is solvable with explicit cost at equilibrium, whenever there exists processes $\{z_t^\alpha, s_t^\alpha, q_t^\alpha,\ \alpha \in [0,1],\ t \in [0,\infty)\} \subset C_b([0,\infty)) \times L^2([0,1])$ that are solutions to the following ODEs:

$$\frac{dz_t^\alpha}{dt} = \Big(a - \frac{b^2}{r}\pi\Big) z_t^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_t^\beta d\beta, \quad (17)$$
$$\frac{ds_t^\alpha}{dt} = \Big(-a + \frac{b^2}{r}\pi + \rho\Big) s_t^\alpha + z_t^\alpha, \quad (18)$$
$$\frac{dq_t^\alpha}{dt} = -\sigma^2\pi + \frac{b^2}{r}(s_t^\alpha)^2 + \rho q_t^\alpha - (z_t^\alpha)^2, \quad (19)$$
$$z_0^\alpha = \int_0^1 g(\alpha,\beta) m^\beta d\beta.$$

The main difficulty with this result is that we don't know the steady-state information $(z_\infty^\alpha, s_\infty^\alpha, q_\infty^\alpha)$ required to solve the ODEs above. To circumvent this obstacle we apply a technique from [1] which consists in solving for $(z_\infty^\alpha, s_\infty^\alpha, q_\infty^\alpha)$ from a steady state condition in the infinite horizon,

$$0 = \frac{dz_\infty^\alpha}{dt} = \frac{ds_\infty^\alpha}{dt} = \frac{dq_\infty^\alpha}{dt}, \quad \forall \alpha \in [0,1]. \quad (20)$$

This yields the family of algebraic equations indexed by $\alpha \in [0,1]$, given below,

$$0 = \Big(a - \frac{b^2}{r}\pi\Big) z_\infty^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_\infty^\beta d\beta, \quad (21)$$
$$0 = \Big(-a + \frac{b^2}{r}\pi + \rho\Big) s_\infty^\alpha + z_\infty^\alpha, \quad (22)$$
$$0 = -\sigma^2\pi + \frac{b^2}{r}(s_\infty^\alpha)^2 + \rho q_\infty^\alpha - (z_\infty^\alpha)^2. \quad (23)$$

We proceed to solve these algebraic equations. From the first two equations, we have

$$0 = \Big(a - \frac{b^2}{r}\pi\Big)\left[\Big(a - \frac{b^2}{r}\pi\Big) - \rho\right] s_\infty^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_\infty^\beta d\beta,$$

with $\alpha \in [0,1]$, which is equivalent (with discrepancies on at most a set of measure zero) to

$$\left[\Big(a - \frac{b^2}{r}\pi\Big)\Big(a - \frac{b^2}{r}\pi\ - \rho\Big) I - \frac{b^2}{r} g\right] \circ s_\infty = 0 \quad (24)$$

where $(g \circ s_\infty)(\cdot) := \int_0^1 g(\cdot,\beta) s_\infty(\beta) d\beta$, and I denotes the identity operator from $L^2([0,1])$ to $L^2([0,1])$.

The operator $\left(\Big(a - \frac{b^2}{r}\pi\Big)\Big(a - \frac{b^2}{r}\pi\ - \rho\Big) I - \frac{b^2}{r} g\right)$ has a bounded inverse if $\frac{r}{b^2}\Big(a - \frac{b^2}{r}\pi\Big)\Big(a - \frac{b^2}{r}\pi\ - \rho\Big)$ is nonzero and not an eigenvalue of the graphon operator g.

Remark 1. *Since it is assumed that $|g(x,y)| \le 1$, for all $x, y \in [0,1]$, the operator norm of g satisfies that*

$$\|g\|_{\text{op}} := \sup_{v \in L^2[0,1]} \frac{\|gv\|}{\|v\|} \le \|g\|_2 \le 1, \text{ (see [20, Lem. 7])}$$

which implies that the absolute values of all the eigenvalues of g are less than or equal to 1. When $a = 0$, from (10), $\pi\big(\frac{b^2}{r}\pi + \rho\big) I - g = I - g$. It has a bounded inverse when 1 is not an eigenvalue of g.

Assumption (A1): The spectrum of the graphon operator g does not cotain

$$\Big(\frac{b^2}{r}\Big)^{-1}\Big(a - \frac{b^2}{r}\pi\Big)\Big(a - \frac{b^2}{r}\pi\ - \rho\Big),$$

where

$$\pi = \sqrt{\frac{r^2(\rho - 2a)^2}{4b^4} + \frac{r}{b^2}} - \frac{(\rho - 2a) r}{2b^2} > 0. \quad (25)$$

Under **Assumption (A1)**, the functional equation (24) admits the (unique) solution in $L^2([0,1])$

$$z_\infty^\alpha = 0 = s_\infty^\alpha, \quad a.e.\ \alpha \in [0,1], \quad (26)$$

and an application of (23) yields

$$q_\infty^\alpha = \frac{\sigma^2\pi}{\rho}, \quad a.e.\ \alpha \in [0,1]. \quad (27)$$

We are interested in calculating an explicit solution, $\{z_t^\alpha, s_t^\alpha, q_t^\alpha, \alpha \in [0,1],\ t \in [0,\infty)\} \subset C_b([0,\infty)) \times L^2([0,1])$, to the following ODEs:

$$\frac{dz_t^\alpha}{dt} = \left(a - \frac{b^2}{r}\pi\right) z_t^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_t^\beta d\beta, \quad (28)$$
$$\frac{ds_t^\alpha}{dt} = \left(-a + \frac{b^2}{r}\pi + \rho\right) s_t^\alpha + z_t^\alpha, \quad (29)$$
$$\frac{dq_t^\alpha}{dt} = -\sigma^2\pi + \frac{b^2}{r}(s_t^\alpha)^2 + \rho q_t^\alpha - (z_t^\alpha)^2, \quad (30)$$
$$z_0^\alpha = \int_0^1 g(\alpha,\beta) m^\beta d\beta.$$

with the infinite horizon conditions

$$z_\infty^\alpha = 0 = s_\infty^\alpha, \quad q_\infty^\alpha = \frac{\sigma^2\pi}{\rho}, \quad a.e.\ \alpha \in [0,1]. \quad (31)$$

Assumption (A2a) The graphon g is of finite rank, that is, there exists $L < \infty$ such that

$$g(\alpha,\beta) = \sum_{\ell=1}^{L} \lambda_\ell f_\ell(\alpha) f_\ell(\beta),$$

where f_ℓ is the orthonormal eigenfunction associated with the non-zero eigenvalue λ_ℓ of g for all $\ell \in \{1, ..., L\}$.

Assumption (A2b) The nonzero eigenvalues $\{\lambda_\ell\}_{\ell=1}^{L}$ of the graphon g satisfy the following bound

$$\lambda_\ell < 1 + \frac{r}{b^2} a(a-\rho),\ \forall \ell \in \{1, \dots, L\}. \tag{32}$$

Assumption (A2c) The following inequality holds:

$$a\sqrt{(\rho-2a)^2 + 4\frac{b^2}{r}} > a(\rho-2a) - \frac{2b^2}{r}. \tag{33}$$

Assumptions (A2b)-(A2c) are introduced to ensure that the equations (29) and (30) have well-defined solutions over the infinite time horizon $[0,\infty)$. Assumption (A2b) is to ensure a crucial second order ODE (41) (to be introduced) has a nonoscillating and exponentially stable solution, and Assumption (A2c) ensures the positivity of (39) (to be introduced later). We note that when $a = 0$ Assumption (A2c) always holds and Assumption (A2b) holds if g does not have 1 as eigenvalue.

Proposition 4. *Let Assumptions (A2a)-(A2b)-(A2c) be in force. Then, the process $\{z_t^\alpha, s_t^\alpha\ \alpha \in [0,1],\ t \in [0,\infty)\}$ is explicitly given as below $\forall t \geq 0,\ a.s.\ \alpha \in [0,1]$,*

$$z_t^\alpha = \sum_{l=1}^{L} f_\ell(\alpha) z_t^\ell, \tag{34}$$

$$s_t^\alpha = -\sum_{l=1}^{L} f_\ell(\alpha) \frac{z_t^\ell}{\big(\theta(\lambda_\ell)+\theta(0)\big)},$$

where

$$z_t^\ell = \lambda_\ell \langle m, f_\ell\rangle \exp\left[\Big(\frac{\rho}{2} - \theta(\lambda_\ell)\Big)t\right],\ \ell \in \{1, \dots, L\}, \tag{35}$$

and $\theta(\cdot)$ is a function defined by

$$\theta(\tau) := \sqrt{\frac{(\rho-2a)^2}{4} + (1-\tau)\frac{b^2}{r}}, \quad \tau \in R. \tag{36}$$

Proof. Consider the graphon spectral decomposition under the finite rank assumption (A2a),

$$g(\alpha,\beta) = \sum_{\ell=1}^{L} \lambda_\ell f_\ell(\alpha) f_\ell(\beta), \quad \forall \alpha,\beta \in [0,1], \tag{37}$$

or equivalently written as

$$g = \sum_{\ell=1}^{L} \lambda_\ell f_\ell f_\ell^T, \quad f_\ell \in L^2([0,1]),$$

where f_ℓ is the orthonormal eigenfunction of g, and λ_ℓ is the eigenvalue associated with f_ℓ. By the definition of eigenvalues and eigenfunctions,

$$g f_\ell = \lambda_\ell f_\ell.$$

Following the spectral reformulation of two point boundary value problems developed in [21], we define the eigen processes

$$z_t^\ell = \langle z_t, f_\ell\rangle, \quad s_t^\ell = \langle s_t, f_\ell\rangle, \quad t \in [0,\infty),\ \ell \in \{1,2,...\}.$$

These processes are solutions to the following equations:

$$\frac{dz_t^\ell}{dt} = \left(a - \frac{b^2\pi}{r}\right) z_t^\ell - \lambda_\ell \frac{b^2}{r} s_t^\ell, \quad z_0^\ell = \lambda_\ell \langle m, f_\ell\rangle,$$
$$\frac{ds_t^\ell}{dt} = z_t^\ell + \left(-a + \frac{b^2\pi}{r} + \rho\right) s_t^\ell, \quad s_\infty^\ell = 0,$$

for which we seek an explicit solution that is compatible with the infinite horizon condition $z_\infty^\ell = 0$, for all $\ell \in \{1, \dots, L\}$. From the ODE for s^ℓ, it admits the representation below:

$$s_t^\ell = -\int_t^\infty \exp\left(\left(-a + \frac{b^2\pi}{r} + \rho\right)(t-s)\right) z_s^\ell ds. \tag{38}$$

The Riccati equation (10) allows to deduce that

$$\left(-a + \frac{b^2\pi}{r} + \rho\right) = a + \frac{1}{\pi}\ , \tag{39}$$

which can be shown to be strictly positive under assumption (A2c) and thus implies that $s_\infty^\ell = 0$.

By substituting this expression for s^ℓ back into the ODE for z^ℓ, we obtain the representation below

$$\frac{dz_t^\ell}{dt} = \left(a - \frac{b^2\pi}{r}\right) z_t^\ell \tag{40}$$
$$+ \lambda_\ell \frac{b^2}{r} \int_t^\infty \exp\left(\left(-a + \frac{b^2\pi}{r} + \rho\right)(t-s)\right) z_s^\ell ds.$$

By differentiating the above ODE and making appropriate substitutions, we obtain the second order ODE for z^ℓ,

$$\frac{d^2 z_t^\ell}{dt} - \rho \frac{dz_t^\ell}{dt}$$
$$+ \left[\lambda_\ell \frac{b^2}{r} - \left(a - \frac{b^2\pi}{r}\right)^2 + \rho\left(a - \frac{b^2\pi}{r}\right)\right] z_t^\ell = 0. \tag{41}$$

Its characteristic equation

$$\xi_\ell^2 - \rho\xi_\ell + \left[-a^2 + \rho a + \frac{b^2}{r}(\lambda_\ell - 1)\right] = 0 \tag{42}$$

admits a solution

$$\xi_\ell = \left(\frac{\rho}{2} - \sqrt{\frac{(\rho-2a)^2}{4} + \frac{b^2}{r}(1-\lambda_\ell)}\ \right),$$
$$= \frac{\rho}{2} - \theta(\lambda_\ell), \tag{43}$$

where $\theta(\lambda_\ell) = \sqrt{\frac{(\rho-2a)^2}{4} + (1-\lambda_\ell)\frac{b^2}{r}}$ is real if $\lambda_\ell < 1 + \frac{r(\rho-2a)^2}{4b^2}$. We observe that

$$\xi_\ell < 0 \text{ if and only if } \lambda_\ell < 1 + \frac{r}{b^2} a(a-\rho).$$

It also holds that,

$$\lambda_\ell < 1 + \frac{r}{b^2} a(a-\rho) \text{ implies } \lambda_\ell < 1 + \frac{r}{b^2} \frac{(\rho-2a)^2}{4},$$

Therefore, whenever assumption (A2b) holds, $\theta(\lambda_\ell)$ is real and $\xi_\ell < 0$ and we obtain z^ℓ as below

$$z_t^\ell = \lambda_\ell \langle m, f_\ell\rangle \exp(\xi_\ell t), \ \forall t \geq 0, \tag{44}$$

where, because $\xi_\ell < 0$ for all $l \in \{1, \ldots, L\}$, the infinite horizon condition $z_\infty^\ell = 0$ is satisfied.

We now proceed to calculate s^ℓ as below, $\forall t \in [0, \infty)$

$$\begin{aligned} s_t^\ell &= -\int_t^\infty \exp\left(\left(-a + \frac{b^2\pi}{r} + \rho\right)(t-s)\right) z_s^\ell ds \quad (45)\\ &= -\lambda_\ell\langle m, f_\ell\rangle \exp\left(\left(-a + \frac{b^2\pi}{r} + \rho\right) t\right) \\ &\quad \int_t^\infty \exp\left(\left(\xi_\ell + a - \frac{b^2\pi}{r} - \rho\right) s\right) ds, \end{aligned}$$

we explicitly calculate the integral, by observing that

$$\xi_\ell < 0, \quad \text{and} \quad \left(a - \frac{b^2\pi}{r} - \rho\right) < 0, \tag{46}$$

and obtain that

$$s_t^\ell = \lambda_\ell\langle m, f_\ell\rangle \exp(\xi_\ell t)\left(\xi_\ell + a - \frac{b^2\pi}{r} - \rho\right)^{-1}. \tag{47}$$

Also, for all $\ell \in \{1, \ldots, L\}$, we have that

$$\begin{aligned} \xi_\ell + a - \frac{b^2\pi}{r} - \rho &= \frac{\rho}{2} - \theta(\lambda_\ell) + a - \frac{b^2\pi}{r} - \rho \\ &= -\theta(\lambda_\ell) + a - \frac{\rho}{2} - \frac{b^2\pi}{r} \\ &= -\theta(\lambda_\ell) - \left(\frac{(\rho-2a)^2}{4} + \frac{b^2}{r}\right)^{\frac{1}{2}} \\ &= -\theta(\lambda_\ell) - \theta(0). \end{aligned}$$

Therefore, it holds that

$$s_t^\ell = -\frac{z_t^\ell}{\theta(\lambda_\ell) + \theta(0)}, \quad \forall \ell \in \{1, \ldots, L\}. \tag{48}$$

Based on (37) and the definition of the eigen processes, we can now reconstruct the solution $\{z_t^\alpha, s_t^\alpha \ \alpha \in [0,1], \ t \in [0,\infty)\}$ as below

$$z_t^\alpha = \sum_{l=1}^L f_\ell(\alpha) z_t^\ell, \quad s_t^\alpha = -\sum_{l=1}^L f_\ell(\alpha)\frac{z_t^\ell}{\theta(\lambda_\ell) + \theta(0)}.$$

where for all $t \geq 0$ and for almost all $\alpha \in [0,1]$. □

Proposition 5. *Let Assumptions (A1)-(A2) be in force. Then, the cost at equilibrium is explicitly given, for almost every $\alpha \in [0,1]$, below*

$$\begin{aligned} J(u^\alpha, z) &= \pi\nu^2 + \pi(m^\alpha)^2 + \frac{\sigma^2\pi}{\rho} - 2m^\alpha \sum_{l=1}^L f_\ell(\alpha)\bar{\lambda}_\ell\langle m, f_\ell\rangle \\ &+ \frac{1}{\rho}\left(\sum_{\ell=1}^L f_\ell(\alpha)\lambda_\ell\langle m, f_\ell\rangle\right)^2 - \frac{b^2}{r\rho}\left(\sum_{\ell=1}^L f_\ell(\alpha)\bar{\lambda}_\ell\langle m, f_\ell\rangle\right)^2 \\ &- \sum_{k=1}^L\sum_{\ell=1}^L f_k(\alpha) f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle\left(\frac{\rho}{2} - \theta(\lambda_k)\right) \\ &\quad \left(\frac{1}{\theta(\lambda_\ell) + \theta(\lambda_k)}\right)\left[\frac{2}{\rho}\lambda_k\lambda_\ell - \frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right], \end{aligned}$$

where we define,

$$\bar{\lambda}_\ell := \frac{\lambda_\ell}{\theta(\lambda_\ell) + \theta(0)}, \quad \ell \in \{1, \ldots, L\}. \tag{49}$$

Proof. Given the process $\{z_t^\alpha, s_t^\alpha \ \alpha \in [0,1], \ t \in [0,\infty)\}$ explicitly calculated for almost every $\alpha \in [0,1]$, we proceed to calculate explicitly the process $\{q_t^\alpha, \ \alpha \in [0,1], \ t \in [0,\infty)\}$, for almost every $\alpha \in [0,1]$.

A straightforward calculation allows to verify that,

$$q_t^\alpha = -\exp(\rho t)\int_t^\infty \Theta(\alpha, s)\exp(-\rho s)\, ds, \tag{50}$$

with $\Theta(\alpha, t), \ \forall \alpha \in [0,1], \ t \in [0,\infty)$, defined by:

$$\begin{aligned} \Theta(\alpha, t) &= -\sigma^2\pi - (z_t^\alpha)^2 + \frac{b^2}{r}(s_t^\alpha)^2, \\ &= -\sigma^2\pi - \left(\sum_{l=1}^L f_\ell(\alpha)\lambda_\ell\langle m, f_\ell\rangle\exp(\xi_\ell t)\right)^2 \\ &+ \frac{b^2}{r}\left(\sum_{\ell=1}^L (\theta(\lambda_\ell) + \theta(0))^{-1} f_\ell(\alpha)\lambda_\ell\langle m, f_\ell\rangle\exp(\xi_\ell t)\right)^2, \\ &= -\sigma^2\pi - \left(\sum_{l=1}^L f_\ell(\alpha)\lambda_\ell\langle m, f_\ell\rangle\exp(\xi_\ell t)\right)^2 \\ &+ \frac{b^2}{r}\left(\sum_{\ell=1}^L f_\ell(\alpha)\bar{\lambda}_\ell\langle m, f_\ell\rangle\exp(\xi_\ell t)\right)^2, \end{aligned}$$

is a solution to the offset ODE,

$$\frac{dq_t^\alpha}{dt} = -\sigma^2\pi + \frac{b^2}{r}(s_t^\alpha)^2 + \rho q_t^\alpha - (z_t^\alpha)^2. \tag{51}$$

Moreover, the process $\{q_t^\alpha, \ \alpha \in [0,1], \ t \in [0,\infty)\}$ is compatible with the infinite horizon condition

$$q_\infty^\alpha = \frac{\sigma^2\pi}{\rho}. \tag{52}$$

Indeed, by applying L'Hopital's Rule, we obtain that,

$$\begin{aligned} \lim_{t\to\infty} q_t^\alpha &= \lim_{t\to\infty} -\exp(\rho t)\int_t^\infty \Theta(\alpha, s)\exp(-\rho s)\, ds \\ &= \lim_{t\to\infty}\frac{\Theta(\alpha, t)}{-\rho} = \frac{-\sigma^2\pi}{-\rho} = q_\infty^\alpha. \end{aligned}$$

Recall that the optimal cost is given, for all $\alpha \in [0,1]$, by

$$J(u^\alpha, z) = \pi(\nu^2 + (m^\alpha)^2) + 2s_0^\alpha m^\alpha + q_0^\alpha.$$

To calculate the cost at equilibrium explicitly, for a.e. $\alpha \in [0,1]$, it is enough to calculate the quantities s_0^α, q_0^α.

We obtain that for almost every $\alpha \in [0,1]$,

$$\begin{aligned} s_0^\alpha &= -\sum_{l=1}^L (\theta(\lambda_\ell) + \theta(0))^{-1} f_\ell(\alpha)\lambda_\ell\langle m, f_\ell\rangle, \quad (53)\\ &= -\sum_{l=1}^L f_\ell(\alpha)\bar{\lambda}_\ell\langle m, f_\ell\rangle \ . \quad (54) \end{aligned}$$

And, for almost every $\alpha \in [0,1]$,

$$q_0^\alpha = -\int_0^\infty \Theta(\alpha, s)\exp(-\rho s)\, ds, \tag{55}$$

where $\Theta(\alpha,t),\ \forall\alpha\in[0,1],\ t\in[0,\infty)$, is defined by:

$$\Theta(\alpha,t)=-\sigma^2\pi-\left(\sum_{l=1}^{L}f_\ell(\alpha)\lambda_\ell\langle m,f_\ell\rangle\exp\left(\xi_\ell t\right)\right)^2+\frac{b^2}{r}\left(\sum_{\ell=1}^{L}f_\ell(\alpha)\bar{\lambda}_\ell\langle m,f_\ell\rangle\exp\left(\xi_\ell t\right)\right)^2.$$

Integrating by parts yields

$$q_0^\alpha=-\frac{\Theta(\alpha,0)}{\rho}-\frac{1}{\rho}\int_0^\infty\exp\left(-\rho s\right)\frac{d\Theta(\alpha,s)}{ds}ds.$$

We then calculate that,

$$-\frac{\Theta(\alpha,0)}{\rho}=\frac{\sigma^2\pi}{\rho}+\frac{1}{\rho}\left(\sum_{\ell=1}^{L}f_\ell(\alpha)\lambda_\ell\langle m,f_\ell\rangle\right)^2-\frac{b^2}{r\rho}\left(\sum_{\ell=1}^{L}f_\ell(\alpha)\bar{\lambda}_\ell\langle m,f_\ell\rangle\right)^2,$$

and

$$\begin{aligned}&-\frac{1}{\rho}\int_0^\infty\exp\left(-\rho s\right)\frac{d\Theta(\alpha,s)}{ds}ds\\&=\sum_{k=1}^{L}\sum_{\ell=1}^{L}\xi_k\left(\int_0^\infty e^{(\xi_k+\xi_\ell-\rho)s}ds\right)\\&\qquad f_k(\alpha)f_\ell(\alpha)\langle m,f_k\rangle\langle m,f_\ell\rangle\left[\frac{2}{\rho}\lambda_k\lambda_\ell-\frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right],\end{aligned}$$

we calculate the exponential integral and obtain,

$$\begin{aligned}&-\frac{1}{\rho}\int_0^\infty\exp\left(-\rho s\right)\frac{d\Theta(\alpha,s)}{ds}ds\\&=\sum_{k=1}^{L}\sum_{\ell=1}^{L}\xi_k\left(\xi_k+\xi_\ell-\rho\right)^{-1}f_k(\alpha)f_\ell(\alpha)\langle m,f_k\rangle\langle m,f_\ell\rangle\\&\qquad\left[\frac{2}{\rho}\lambda_k\lambda_\ell-\frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right].\end{aligned}$$

By observing the equality

$$\left(\xi_k+\xi_\ell-\rho\right)=-\left(\theta(\lambda_\ell)+\theta(\lambda_k)\right),$$

we deduce that,

$$\begin{aligned}q_0^\alpha=&\frac{\sigma^2\pi}{\rho}+\frac{1}{\rho}\left(\sum_{\ell=1}^{L}f_\ell(\alpha)\lambda_\ell\langle m,f_\ell\rangle\right)^2\\&-\frac{b^2}{r\rho}\left(\sum_{\ell=1}^{L}f_\ell(\alpha)\bar{\lambda}_\ell\langle m,f_\ell\rangle\right)^2\\&-\sum_{k=1}^{L}\sum_{\ell=1}^{L}\left(\frac{\rho}{2}-\theta(\lambda_k)\right)\left(\theta(\lambda_\ell)+\theta(\lambda_k)\right)^{-1}\\&\qquad f_k(\alpha)f_\ell(\alpha)\langle m,f_k\rangle\langle m,f_\ell\rangle\left[\frac{2}{\rho}\lambda_k\lambda_\ell-\frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right].\end{aligned}$$

Finally, recalling that the cost at equilibrium is explicitly given by

$$J(u^\alpha,z)=\pi(\nu^2+(m^\alpha)^2)+2s_0^\alpha m^\alpha+q_0^\alpha \tag{56}$$

and substituting the calculated terms appropriately, we obtain the desired result. □

Remark 2 (Properties of the $\theta(\cdot)$ function)**.** *The $\theta(\cdot)$ function has two interesting properties due to its particular form*

$$\theta(\tau):=\sqrt{\frac{\left(\rho-2a\right)^2}{4}+(1-\tau)\frac{b^2}{r}}.$$

A first property is that for $\lambda_\ell\neq\lambda_k$

$$\frac{1}{\theta(\lambda_\ell)+\theta(\lambda_k)}=\frac{\theta(\lambda_\ell)-\theta(\lambda_k)}{\theta(\lambda_\ell)^2-\theta(\lambda_k)^2}=-\left(\frac{r}{b^2}\right)\frac{\theta(\lambda_\ell)-\theta(\lambda_k)}{(\lambda_\ell-\lambda_k)}. \tag{57}$$

A second property is that $\theta'(\tau)=\frac{-b^2}{2r\theta(\tau)}$.

In the next proposition, we introduce simplifications of the cost at equilibrium calculated above.

Proposition 6. *Assume (A1)-(A2) hold. Then, the cost at equilibrium is explicitly given below: for almost every $\alpha\in[0,1]$,*

$$\begin{aligned}J(u^\alpha,z)=&\pi\nu^2+\pi(m^\alpha)^2+\frac{\sigma^2\pi}{\rho}\\&-\frac{2r}{b^2}m^\alpha\sum_{\ell=1}^{L}f_\ell(\alpha)(\theta(0)-\theta(\lambda_\ell))\langle m,f_\ell\rangle\\&-\sum_{k=1}^{L}\sum_{\ell=1}^{L}f_k(\alpha)f_\ell(\alpha)\langle m,f_k\rangle\langle m,f_\ell\rangle\left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)}-2\right)\\&\qquad\frac{1}{\rho}\Big[\lambda_\ell\lambda_k-\frac{r}{b^2}(\theta(0)-\theta(\lambda_\ell))(\theta(0)-\theta(\lambda_k))\Big],\end{aligned}$$

where $\theta(\tau):=\sqrt{\frac{(\rho-2a)^2}{4}+(1-\tau)\frac{b^2}{r}},\ \tau\in R$.

Proof. We observe that

$$\begin{aligned}&\sum_{k=1}^{L}\sum_{\ell=1}^{L}f_k(\alpha)f_\ell(\alpha)\langle m,f_k\rangle\langle m,f_\ell\rangle\left(\frac{\rho}{2}-\theta(\lambda_k)\right)\\&\qquad\left(\frac{1}{\theta(\lambda_\ell)+\theta(\lambda_k)}\right)\left[\frac{2}{\rho}\lambda_k\lambda_\ell-\frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right]\\&=\sum_{k=1}^{L}\sum_{\ell=1}^{L}f_k(\alpha)f_\ell(\alpha)\langle m,f_k\rangle\langle m,f_\ell\rangle\\&\qquad\left(\frac{\rho-\theta(\lambda_k)-\theta(\lambda_\ell)}{\theta(\lambda_\ell)+\theta(\lambda_k)}\right)\left[\frac{1}{\rho}\lambda_k\lambda_\ell-\frac{b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right]\\&=\sum_{k=1}^{L}\sum_{\ell=1}^{L}f_k(\alpha)f_\ell(\alpha)\langle m,f_k\rangle\langle m,f_\ell\rangle\\&\quad\left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)}\right)\left[\frac{1}{\rho}\lambda_k\lambda_\ell-\frac{b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right]\\&-\frac{1}{\rho}\left(\sum_{\ell=1}^{L}f_\ell(\alpha)\lambda_\ell\langle m,f_\ell\rangle\right)^2+\frac{b^2}{r\rho}\left(\sum_{\ell=1}^{L}f_\ell(\alpha)\bar{\lambda}_\ell\langle m,f_\ell\rangle\right)^2.\end{aligned}$$

Taking the cost form in Prop. 5, then last three terms there can be further simplified, which leads to the following result

$$J(u^\alpha, z) = \pi\nu^2 + \pi(m^\alpha)^2 + \frac{\sigma^2\pi}{\rho} - 2m^\alpha \sum_{l=1}^{L} f_\ell(\alpha)\bar{\lambda}_\ell \langle m, f_\ell\rangle$$
$$-\sum_{k=1}^{L}\sum_{\ell=1}^{L} f_k(\alpha) f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle$$
$$\left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)} - 2\right)\left[\frac{1}{\rho}\lambda_k\lambda_\ell - \frac{b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right]. \tag{58}$$

An application of the property (57) yields

$$\bar{\lambda}_\ell := \frac{\lambda_\ell}{\theta(\lambda_\ell)+\theta(0)} = -\frac{r}{b^2}(\theta(\lambda_\ell) - \theta(0)),\ \ \ell \in \{1,\dots,L\}.$$

Replacing $\bar{\lambda}_k$ and $\bar{\lambda}_\ell$ in (58) yields the desired result. □

Assumption (A3) The initial means are constant across all nodes. That is, for all $\alpha \in [0,1]$,

$$m^\alpha = m, \quad \text{for some} \quad m \in R. \tag{59}$$

Proposition 7. *Assume that (A1)-(A2)-(A3) hold. The cost at equilibrium admits the following representation, for almost every $\alpha \in [0,1]$,*

$$J(u^\alpha, z) = \pi\left(\nu^2 + m^2 + \frac{\sigma^2}{\rho}\right) - \frac{2r}{b^2}m^2\int_0^1 \hat{g}(\alpha,\beta)d\beta$$
$$- m^2\int_0^1 \tilde{g}(\alpha,\beta\ |\alpha)d\beta,$$

where the introduced finite rank graphons $\{\hat{g}(\cdot,\cdot),\ \tilde{g}(\cdot,\cdot\ |\ \alpha),\ \forall\alpha \in [0,1]\}$ are defined for all $(\epsilon,\beta) \in [0,1]\times[0,1]$ by

$$\hat{g}(\epsilon,\beta) := \sum_{k=1}^{L}\hat{\lambda}_k f_k(\epsilon) f_k(\beta), \tag{60}$$
$$\tilde{g}(\epsilon,\beta\ |\alpha) := \sum_{k=1}^{L}\tilde{\lambda}_k^\alpha f_k(\epsilon) f_k(\beta), \tag{61}$$

and for all $k \in \{1,\dots,L\}$, for all $\alpha \in [0,1]$, the eigenvalues are defined by

$$\hat{\lambda}_k = \theta(0) - \theta(\lambda_k),$$

$$\tilde{\lambda}_k^\alpha := \sum_{\ell=1}^{L} f_\ell(\alpha)\langle 1, f_\ell\rangle\left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)} - 2\right)$$
$$\frac{1}{\rho}\left(\lambda_k\lambda_\ell - \frac{r}{b^2}\hat{\lambda}_k\hat{\lambda}_\ell\right).$$

Proof. Thanks to assumptions (A1)-(A2) and proposition 6, we have that the cost at equilibrium is given, for almost every $\alpha \in [0,1]$, by

$$J(u^\alpha, z) = \pi\nu^2 + \pi(m^\alpha)^2 + \frac{\sigma^2\pi}{\rho}$$
$$- \frac{2r}{b^2}m^\alpha \sum_{l=1}^{L} f_\ell(\alpha)\hat{\lambda}_\ell\langle m, f_\ell\rangle$$
$$- \sum_{k=1}^{L}\sum_{\ell=1}^{L} f_k(\alpha) f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle$$
$$\left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)} - 2\right)\frac{1}{\rho}\left[\lambda_k\lambda_\ell - \frac{r}{b^2}\hat{\lambda}_k\hat{\lambda}_\ell\right].$$

Assuming that (A3) hold, we get

$$J(u^\alpha, z) = \pi\left(\nu^2 + m^2 + \frac{\sigma^2}{\rho}\right) - \frac{2r}{b^2}m^2\sum_{l=1}^{L} f_\ell(\alpha)\hat{\lambda}_\ell\langle 1, f_\ell\rangle$$
$$- m^2\sum_{k=1}^{L}\tilde{\lambda}_k^\alpha f_k(\alpha)\langle 1, f_k\rangle,$$

where for all $k \in \{1,\dots,L\}$, for all $\alpha \in [0,1]$, the quantities $\bar{\lambda}_k,\ \tilde{\lambda}_k^\alpha$, are defined by

$$\hat{\lambda}_k = \theta(0) - \theta(\lambda_k),$$

$$\tilde{\lambda}_k^\alpha := \sum_{\ell=1}^{L} f_\ell(\alpha)\langle 1, f_\ell\rangle\left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)} - 2\right)$$
$$\frac{1}{\rho}\left(\lambda_k\lambda_\ell - \frac{r}{b^2}\hat{\lambda}_k\hat{\lambda}_\ell\right).$$

Interpreting these quantities as eigenvalues, we deduce that the cost at equilibrium can be written as a function of the degrees of newly introduced finite rank graphons build from the original graphon $g(\cdot,\cdot)$. □

The next proposition gives a necessary and sufficient condition for a node $\alpha^* \in [0,1]$ to be, almost surely, a node with minimal cost at equilibrium.

Proposition 8. *Assume that (A1)-(A2)-(A3) hold. Any node $\alpha^* \in [0,1]$ is, almost surely, a node with minimal cost at equilibrium, if and only if, $\alpha^* \in [0,1]$ satisfies the condition:*

$$\alpha^* = \mathit{argmax}_{\alpha\in[0,1]}\left[\frac{2r}{b^2}\int_0^1 \hat{g}(\alpha,\beta)d\beta + \int_0^1 \tilde{g}(\alpha,\beta\ |\alpha)d\beta\right] \tag{62}$$

Proof. The proof is straightforward from the observation that, by proposition 7, the cost at equilibrium can be written as

$$J(u^\alpha, z) = \pi\left(\nu^2 + m^2 + \frac{\sigma^2}{\rho}\right)$$
$$- m^2\left[\frac{2r}{b^2}\int_0^1 \hat{g}(\alpha,\beta)d\beta + \int_0^1 \tilde{g}(\alpha,\beta\ |\alpha)d\beta\right].$$

□

Remark 3. *We note that, whenever $\alpha^* \in [0,1]$ satisfying 62 is an interior point of $[0,1]$, it holds that*

$$\frac{\partial J(u^{\alpha^*}, z)}{\partial\alpha} = 0, \tag{63}$$

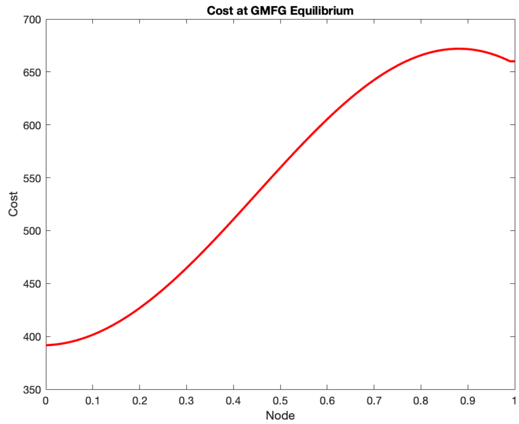

Fig. 1: Cost at equilibrium as a function of $\alpha \in [0, 1]$.

thus linking $\alpha^* \in [0, 1]$ to the notion of critical nodes for LQG-GMFGs introduced in [10]. Therein the uniform attachment graphon is used as an example and it is not of finite rank. This difficulty can be solved by using some finite rank approximation of its spectral decomposition (see [12]). As an illustration, Fig. 1 represents the cost at equilibrium when we consider the 1-rank approximation of the uniform attachment graphon given by the eigenvalue and eigenfunction below

$$\lambda = (2/3.14)^2, \quad f(\alpha) = \sqrt{2}\cos\left(\frac{3.14 \times \alpha}{2}\right), \ \alpha \in [0, 1],$$

and the infinite horizon LQG-GMFGs with $a = \rho = 0.5, \nu = 1, \sigma = 0.15, b = 1, m = 10$.

Remark 4. *Note that the differential calculus for GMFGs with respect to the nodes is made rigorous via graphon vertex embedding in some compact subset of R^d with $d \geq 1$ which is possible due to the work [22], and hence is one future direction of this paper.*

III. Conclusion

In this work, we establish the explicit form of equilibrium cost for infinite horizon LQG-GMFGs. This allows us to deduce a necessary and sufficient condition for identifying nodes, $\alpha \in [0, 1]$, associated with minimal equilibrium cost. These conditions are structural and involve new graphons built from the original graphon in the infinite horizon LQG-GMFG. In future works, we will further analyze these new graphons, the relaxation of the finite-rank assumption on graphons for infinite horizon LQG-GMFGs following similar ideas in [12], and possible relations to centrality notions in games on large networks (see [23], [24]).

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