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Critical Nash Value Nodes for Control Affine Embedded Graphon Mean Field Games ⋆

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Abstract: Graphon Mean Field Games (GMFGs) (Caines and Huang (2021)) constitute generalizations of Mean Field Games to the case where the agents form subpopulations associated with the nodes of large graphs. The work in (Foguen-Tchuendom et al. (2021), Foguen-Tchuendom et al. (2022a)) analyzed the stationarity of equilibrium Nash values with respect to node location for large populations of non-cooperative agents with linear dynamics on large graphs embedded in Euclidean space together with their limits (termed embedded graphons). That analysis is extended in this investigation to agent systems lying in the class of control affine non-linear systems (see Isidori (1985)). Specifically, control affine GMFG systems are treated where (i) at each node $\alpha \in V$ the drift of each generic agent system is affine in the control function, and (ii) the running costs at each node $\alpha \in V \subset R^m$ are exponentiated negative inverse quadratic (ENIQ) functions of the difference between a generic state and the local graphon weighted mean Z^{α,μ_G} where $\mu_G := \{\mu_\beta, \beta \in V \subset R^m\}$ is the globally distributed family of mean fields. The infinite cardinality node and edge limits are considered, where it is assumed that the limit embedded graphon $g(\alpha, \beta)$, $(\alpha, \beta) \in V \times V$, is continuously differentiable. It is shown that the equilibrium Nash value V^α is stationary with respect to the nodal location $\alpha \in V$ if and only if the corresponding mean Z^{α,μ_G} is stationary with respect to nodal location.

Keywords: Mean Field Games, Networks, Graphons

1. INTRODUCTION

The literature on Mean Field Games on graphons is growing, see for example Caines and Huang (2019), Caines and Huang (2021), Lacker and Soret (2022), Delarue (2017), Parise and Ozdaglar (2023), Carmona et al. (2022). The models used in this work are generalizations those used in standard Mean Field Game theory (see e.g. Carmona and Delarue (2018a,b)), where the agents are essentially coupled on complete graphs with uniform weights. Our study is set in the framework known as Graphon Mean Field Game theory, see Caines and Huang (2019), Caines and Huang (2021). Equipped with the latter theory, we continue the investigation of the existence and properties of what are termed critical nodes (i.e. stationary Nash value nodes) for games involving large populations of agents distributed over large networks. Initially, Foguen-Tchuendom et al. (2021, 2022a) analyzed the stationarity of equilibrium Nash values with respect to node location for large populations of non-cooperative agents controlling linear quadratic Gaussian (LQG) systems on large graphs embedded in Euclidean space together with their limits, termed embedded graphons. As a follow up, we study the link between the optimality of nodes and their degrees in the network Foguen-Tchuendom et al. (2022b). The initial analysis is extended in this investigation to agent systems lying in the class of control affine non-linear systems (see Isidori (1985)).

Consider models of large population games, for which the N agents $\mathcal{A}_i, 1 \le i \le N < \infty$, are distributed over the finite network, represented by the graph G_k defined by its adjacency matrix $(g^k_{i,j})_{i,j=1:M_k}$. We assume that, at each node of this graph, there is a cluster of agents and let $\mathbf{X}_{G_k} = \bigoplus_{l=1}^{M_k}\{X^i | i \in C_l\}$ denote the states of all agents in the total set of clusters of the population. Hence $N = \sum_{l=1}^{M_k} |C_l|$. All spatially distributed clusters lie at the nodes of the graph G_k and interact via the weighted averages (1) defined by the finite graph G_k. For each agent $\mathcal{A}_i$, whose cluster is denoted by $C(i)$, the coupling term

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(also called the local graphon weighted mean field term) governing its interaction with other players via the network is given by:

$$Z_t^{i,G_k} = \frac{1}{M_k}\sum_{l=1}^{M_k} g_{C(i),l}^k \frac{1}{|C_l|}\sum_{j\in C_l} X_t^j,\ \forall t\in[0,T]. \qquad (1)$$

The specification of $\{Z_t^{i,G_k}; t \in [0,T]\}$ relies on the sectional information $g_{i,\bullet}^k$ of $\mathcal{A}_i$. All the individuals residing in cluster C_l, including the agent's local cluster C_i, are symmetric and their average generates an overall impact on each agent $\mathcal{A}_i$ in the ith cluster via the local graphon weighted mean field term a shown in (2) below.

The state evolution of the collection of N agents $\mathcal{A}_i, 1 \le i \le N < \infty$, is specified by a set of N control affine stochastic differential equations (SDEs) over a finite horizon of duration $0 < T < \infty$. For each agent $\mathcal{A}_i$, at some node the state evolution is given by

$$\begin{aligned} &dX_t^i = \big(a(X_t^i) + b(X_t^i)u_t^i + c(X_t^i)Z_t^{i,G_k}\big)dt + \sigma dW_t^i, \\ &X_0^i \sim \mathcal{N}(m, v^2), \qquad \forall t\in[0,T], \end{aligned} \qquad (2)$$

where $a(\cdot), b(\cdot), c(\cdot)$ are bounded differentiable functions with bounded uniformly Lipschitz continuous differentials, and $\sigma \ge 0$. Here $X_t^i \in \mathbb{R}$ denotes the state, $u_t^i \in \mathbb{R}$ the control input and Z_t^{i,G_k} the local graphon weighted mean field specified in (1). For simplicity, all initial conditions are taken to satisfy, $X_0^i \sim \mathcal{N}(m, v^2),\ v > 0,\ m \in \mathbb{R}$. Let $\{W^i, i = 1, ..., N\}$ denote a collection of independent Brownian motions defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions.

Furthermore, each agent $\mathcal{A}_i$ has a cost given by

$$\begin{aligned} &J_i^N(u^i, u^{-i}) \\ &:= \mathbb{E}\int_0^T \left[\frac{r}{2}(u_t^i)^2 + \exp\Big(-\frac{q}{2}\big(X_t^i - \gamma(t)Z_t^{i,G_k}\big)^{-2}\Big)\right]dt, \end{aligned} \qquad (3)$$

where $1 \le i \le N$, $\gamma(t)$ is a square integrable function of time and u^{-i} denotes the controls of all agents other than $\mathcal{A}_i$. We note that the exponentiated negative inverse quadratic (ENIQ) running cost function on the system state in (3) vanishes at the origin and is strictly positive, monotonically increasing, infinitely differentiable and bounded by unity on $(0, \infty)$.

The above set-up constitutes what is often called a large scale dynamic stochastic games. A notion of solution for these games is the well-known Nash equilibrium.

Definition 1. (Nash Equilibrium). Any collection of controls for the large dynamic stochastic network games denoted $(u^{i*}, i = 1, ..., N)$, is a Nash equilibrium if and only if, any unilateral deviation, from u^{i*} to any other control u^i, yields a higher cost. That is,

$$J_i^N(u^{i*}, u^{-i*}) \le J_i^N(u^i, u^{-i*}),\ \forall i = 1, ..., N. \qquad (4)$$

Finding a Nash equilibrium when both the cluster size and the network size are large would be intractable. However, when the network describing the interaction between the agents is uniform, the theory of Mean Field Games (Huang et al. (2006), Lasry and Lions (2006)) provides a systematic approach to the problem (see the monograph of Carmona and Delarue (2013)).

For non-uniform networks, different formulations have been given to this problem (see e.g. Caines and Huang (2019), Caines and Huang (2021), Lacker and Soret (2022), Delarue (2017), Parise and Ozdaglar (2023), Carmona et al. (2022)) and in the present paper we follow the Graphon Mean Field Games paradigm (Caines and Huang (2019), Caines and Huang (2021)).

In the large scale limit defined here, the number of nodes, M_k, of G_k tends to infinity and the smallest size of clusters at each node, $\min_{l=1:M_k}|C_l|$, tends to infinity, and hence the number of agents, N, also goes to infinity.

We further assume that the sequence of graphs $\{G_k; k \in \mathbb{N}\}$ consists of a sequence of nodes (or vertices), and node pairs, corresponding to edges, which are embedded in the unit m and $2m$-dimensional cubes in $\mathbb{R}^m$ and $\mathbb{R}^{2m}$ respectively. As shown in Caines (2022), such sequences converge in the sense of distribution functions converging at continuity points, or equivalently in terms of the weak convergence of measures. Hence the associated limit objects are taken to be the limiting measures. (Such a construction is inspired by, but is distinct from, that of cut-metric convergence in the standard theory of graphons (see Lovasz (2012)).) Specifically, the embedded graph vertex (respectively graph edge) limit set is a measure on the m-dimensional (respectively $2m$-dimensional) unit cube. This is in contrast to the standard graphon which is a symmetric Lebesgue measurable function $W : [0,1]^2 \to [0,1]$ which can be interpreted as weighted graphs on the vertex set $[0,1]$. The α parameter used in this paper as the node index for the embedded graph limit takes its values in $[0,1]^m$.

For simplicity of analysis shall we assume the limit measures have distribution functions which possess continuously differentiable densities, and, as a general notation for such embedded graphon densities, we write

$$\begin{aligned} g : [0,1]^m \times [0,1]^m &\longrightarrow [0,\infty) \\ (\alpha,\beta) &\mapsto g(\alpha,\beta). \end{aligned}$$

Furthermore, for simplicity of exposition in this paper we assume $m = 1$, and provide an example in which one considers a sequence of uniform attachment graphs (Lovasz, 2012), and obtains the following embedded graphon density (in the limit)

$$\begin{aligned} g : [0,1] \times [0,1] &\longrightarrow [0,1] \\ (\alpha,\beta) &\mapsto g(\alpha,\beta) = 1 - \max\{\alpha,\beta\}, \end{aligned}$$

as illustrated in the figure below

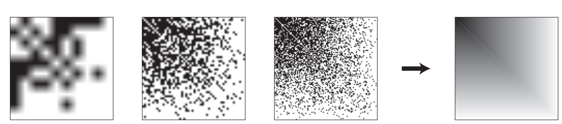

Fig. 1. Graph Sequence Converging to its Limit (Lovasz, 2012)

Parallel to the standard MFG formulation, the infinite population of agents at all graphon nodes, $\alpha \in [0,1]$, admits representative agents, whose state evolution is given by control affine SDEs of the form introduced in (2) above:

$$\begin{aligned} &dX_t^\alpha = \big(a(X_t^\alpha) + b(X_t^\alpha)u_t^\alpha + c(X_t^\alpha)Z_t^{\alpha,g}\big)dt + \sigma dW_t^\alpha, \\ &X_0^\alpha = \xi^\alpha \sim \mathcal{N}(m, v^2), \quad \forall t\in[0,T], \quad \forall \alpha\in[0,1], \end{aligned} \qquad (5)$$

where, the random variables $(\xi^\alpha)_{\alpha\in[0,1]}$ have the same distribution as X_0^0, and the Brownian motions $(W_t^\alpha)_{t\in[0,T]}$, $\alpha \in [0,1]$, have the same distribution as $W_t^0, t \in [0,T]$. Note that no form of stochastic process along the interval $\{\alpha \in [0,1]\}$ is defined in this paper.

In the large scale limit, each representative agent indexed by $\alpha \in [0,1]$ minimizes a cost function given by

$$J(u^\alpha, \mu) := \mathbb{E}\int_0^T \Big[\frac{r}{2}(u_t^\alpha)^2 + \exp\Big(-\frac{q}{2}\big(X_t^\alpha - \gamma(t)Z_t^{\alpha,g}\big)^{-2}\Big)\Big]dt, \tag{6}$$

and at all nodes $\alpha \in [0,1]$, the global mean field term denoted $Z_t^{\alpha,g}, t \in [0,T]$, is defined as

$$Z_t^{\alpha,g} := \int_0^1 g(\alpha,\beta)\int_{\mathbb{R}} y d\mu(\beta,t)(y)d\beta,\ \forall t \in [0,T], \tag{7}$$

where for all $\alpha \in [0,1], t \in [0,1]$, $\mu(\alpha,t)$ is in the set of probability measures with finite second moment, denoted $\mathcal{P}_2(\mathbb{R})$.

2. THE CONTOL AFFINE GMFG PROBLEM AND ITS EQUATIONS

In this section, we formalize and describe the solvability of the Graphon Mean Field Games associated with the control affine model introduced in the previous section.

2.1 Formulation of the GMFG Problem

Define the following admissible control space,

$$\mathbb{A} := \{u : \Omega\times[0,T] \mapsto \mathbb{R} \mid u(\cdot)\ \mathbb{F}-\text{progressively measurable and } \mathbb{E}\Big[\int_0^T |u(t)|^2 dt\Big] < \infty\},$$

and the corresponding instance of a Control Affine (Quadratic Gaussian) Graphon Mean Field Game (CA-GMFG) problem.

Find a two-parameter family of probability measures in $\mathcal{P}_2(\mathbb{R})$, denoted $\mu(\alpha,t)$, $\forall t \in [0,T]$, $\forall \alpha \in [0,1]$, such that:

1) **Agents' Control Problems**:
There exists α-nodal optimal control laws, denoted $u^{\alpha,o} := (u_t^{\alpha,o})_{t\in[0,T]} \in \mathbb{A}$ for all $\alpha \in [0,1]$, such that

$$J(u^{\alpha,o},\mu) = \min_{u^\alpha\in\mathbb{A}} J(u^\alpha,\mu) \tag{8}$$
$$= \min_{u^\alpha\in\mathbb{A}} \mathbb{E}\int_0^T \Big[\frac{r}{2}\big(u_t^\alpha\big)^2 + \exp\Big(-\frac{q}{2}\big(X_t^\alpha - \gamma(t)Z_t^{\alpha,g}\big)^{-2}\Big)\Big]dt$$

subject to the dynamics for all $t \in [0,T]$

$$dX_t^\alpha = \sigma dW_t^\alpha, + \big(a(X_t^\alpha) + b(X_t^\alpha)u_t^\alpha + c(X_t^\alpha)Z_t^{\alpha,g}\big)dt \tag{9}$$

$$Z_t^{\alpha,g} = \int_0^1 g(\alpha,\beta)\int_{\mathbb{R}} v d\mu(\beta,t)(v)d\beta, \tag{10}$$

with $X_0^\alpha = \xi^\alpha \sim \mathcal{N}(m,v^2), m, v \in \mathbb{R}, v^2 > 0$.

2) **Consistency Conditions**:
The optimal state trajectories $(X_t^{\alpha,\mu,o})_{t\in[0,T]}, \forall\alpha \in [0,1]$, generated in Part 1) satisfy the GMFG McKean-Vlasov consistency conditions:

$$\mu(\alpha,t) = \mathcal{L}\big(X_t^{\alpha,\mu,o}\big), \quad \forall(\alpha,t) \in [0,1]\times[0,T]. \tag{11}$$

2.2 Solvability of the Control Affine-GMFG Problem

The analysis in this section establishes that one can solve the Control Affine GMFG problem via the resolution of a system of Forward Backward Partial Differential Equations (FBPDEs) describing the value function and probability density function of agents involved in the Control Affine GMFG problem.

We proceed in a two step approach. Firstly, by fixing probability density functions for the states of the representative agents we derive the Hamilton-Jacobi-Bellman (HJB) equations for their value functions together with the terminal conditions. Secondly, given the resulting control laws for the representative agents, we derive the Fokker-Kolmogorov-Planck (FKP) equations for their probability density functions together with initial conditions. Subject to the consistency condition on the generated density functions, these two coupled sets of equations constitute the entire Controlled Affine GMFG system.

Concerning existence and uniqueness of the solutions to the derived Control Affine GMFG equations, we note that assumptions on the functions and running costs in (5), (6), (7), including specified bounds on the Lipschitz coefficients, are used in Caines and Huang (2021) to obtain via a Banach contraction argument the existence and uniqueness of solutions to GMFG equations more general than the following Control Affine GMFG equations.

HJB Equations

We introduce, for all $(\alpha,t,x) \in [0,1]\times[0,T]\times\mathbb{R}$ the probability density functions $p(\alpha,t,x)$ satisfying the condition

$$d\mu(\alpha,t)(x) = p(\alpha,t,x)dx,$$

and we define the systems' Hamiltonians in terms of the notation introduced above, namely,

$$H\Big[t,x,\frac{\partial V(\alpha,t,x)}{\partial x},Z,u\Big] := \big(a(x)+b(x)u+c(x)Z\big)\frac{\partial V(\alpha,t,x)}{\partial x} + \Big[\frac{r}{2}u^2 + \exp\Big(-\frac{q}{2}\big(x-\gamma(t)Z\big)^{-2}\Big)\Big], \tag{12}$$

with $x,u,q,Z \in \mathbb{R}$, $\gamma(\cdot) \in C(\mathbb{R},\mathbb{R})$, and $V(\alpha,t,x)$ the value functions of the representative agents. Applying the dynamic programming principle, we obtain that the value functions are given as solutions to the HJB equations

$$-\frac{\partial V(\alpha,t,x)}{\partial t} = \inf_{u\in\mathbb{A}} H\Big[t,x,\frac{\partial V(\alpha,t,x)}{\partial x},Z_t^{\alpha,g},u\Big] + \frac{\sigma^2}{2}\frac{\partial^2 V(\alpha,t,x)}{\partial x^2},$$
$$= \Big[\exp\Big(-\frac{q}{2}\big(x-\gamma(t)Z_t^{\alpha,g}\big)^{-2}\Big) - \frac{b^2(x)}{2r}\Big(\frac{\partial V(\alpha,t,x)}{\partial x}\Big)^2 + \Big(a(x)+c(x)Z_t^{\alpha,g}\Big)\Big(\frac{\partial V(\alpha,t,x)}{\partial x}\Big)\Big] + \frac{\sigma^2}{2}\Big(\frac{\partial^2 V(\alpha,t,x)}{\partial x^2}\Big), \quad V(\alpha,T,x)=0, \tag{13}$$

where $Z_t^{\alpha,g}$ is given by

$$Z_t^{\alpha,g} = \int_0^1 g(\alpha,\beta)\int_{\mathbb{R}} vp(\beta,t,v)dvd\beta.$$

FKP Equations

Given the value functions and probability density functions, $\{V(\alpha,t,x), p(\alpha,t,x), (\alpha,t,x) \in [0,1]\times[0,T]\times\mathbb{R}\}$, we obtain the following optimal controls, $\{u_t^{\alpha,o},\ (\alpha,t) \in [0,1]\times[0,T]\}$, and the optimal states, $\{X_t^{\alpha,o},\ (\alpha,t) \in [0,1]\times[0,T]\}$, for the representative agents

$$u_t^{\alpha,o} = -\frac{b(X_t^{\alpha,o})}{r}\frac{\partial V(\alpha,t,X_t^{\alpha,o})}{\partial x}, \quad X_t^{\alpha,o} = \xi^\alpha,$$
$$dX_t^{\alpha,o} = \sigma dW_t^\alpha + \left((a(X_t^{\alpha,o}) + c(X_t^{\alpha,o})Z_t^{\alpha,g} - \frac{b^2}{r}\frac{\partial V(\alpha,t,X_t^{\alpha,o})}{\partial x}\right)dt,$$

and derive the following FKP equations for the probability density functions associated with the SDEs describing the optimal states,

$$\frac{\partial p(\alpha,t,x)}{\partial t} = -\frac{\partial}{\partial x}\left[p(\alpha,t,x)\left(a(x) + c(x)Z_t^{\alpha,g} - \frac{b^2(x)}{r}\frac{\partial V(\alpha,t,x)}{\partial x}\right)\right] + \frac{\sigma^2}{2}\frac{\partial^2 p(\alpha,t,x)}{\partial x^2},$$
$$p(\alpha,0,x) = \frac{1}{\sqrt{2\pi v^2}}\exp\left(-\frac{1}{2}\left(\frac{x-m}{v}\right)^2\right). \tag{14}$$

The coupled FBPDEs (13) and (14) constitute the Control Affine GMFG equations and their solutions are given by

$$\{V(\alpha,t,x), p(\alpha,t,x),\ (\alpha,t,x) \in [0,1]\times[0,T]\times\mathbb{R}\}.$$

3. CRITICAL NODES FOR GMFGS

Recall that the global mean field, $Z_t^{\alpha,g}$, defined by

$$Z_t^{\alpha,g} = \int_0^1 g(\alpha,\beta)\int_{\mathbb{R}} vp(\beta,t,v)dvd\beta,$$

is an interaction term describing the influence of the limit network on the dynamics of the representative agents at each node $\alpha \in [0,1]$.

In this section, we consider the particular nodes at which the first derivative of the global graphon mean field with respect to $\alpha \in [0,1]$ vanishes, which we call mean critical nodes. These nodes are well-defined whenever the following assumptions hold.

Assumption A1: There exist unique solutions (V,p) to the Control Affine GMFG equations (13) and (14) and all mixed partial derivatives of V up to order one in time t, two in space x and one in the α variable exist and all are jointly continuous in all the variables $\{t,x,\alpha\}$.

Assumption A2: The embedded graphon function $g(\cdot,\cdot)$ is continuously differentiable a.e on $[0,1]^2$.

Definition 2. (Mean Critical Node). A node $\lambda \in [0,1]$ is a *mean critical node* for a Control Affine GMFG system if the following local mean field stationary condition holds for $Z_t^{\alpha,g}$

$$\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\bigg|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T]. \tag{15}$$

For two particular examples of embedded graphons, one can readily identify mean critical nodes and observe that they coincide with important nodes in the family of graphs whose limits are associated with the embedded graphons as follows:

E1 Consider first the limit graphon of a sequence of finite Erdös-Rényi graphs. Indeed, it is defined as:

$$g(\alpha,\beta) := k \in (0,1),\ \forall(\alpha,\beta) \in [0,1]^2.$$

Then, we can obtain that

$$Z_t^{\alpha,g} = k\int_0^1 \mathbb{E}\big[X_t^{\beta,o}\big]d\beta,\ \forall(\alpha,t) \in [0,1]\times[0,T],$$

from which it follows that, for all $\lambda \in [0,1]$:

$$\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\bigg|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T].$$

That is to say, if the graphon is a limit of Erdös-Rényi finite graphs, then for the associated Control Affine GMFG problem, all nodes $\lambda \in [0,1]$ are mean critical nodes.

E2 Consider second the uniform attachment graphon:

$$g(\alpha,\beta) = 1 - \max\{\alpha,\beta\},\ \forall(\alpha,\beta) \in [0,1]^2.$$

Then, we can compute that for all $(\alpha,t) \in [0,1]\times[0,T]$

$$Z_t^{\alpha,g} = \int_0^1 (1-\max\{\alpha,\beta\})\mathbb{E}\big[X_t^{\beta,o}\big]d\beta, \tag{16}$$
$$= (1-\alpha)\int_0^\alpha \mathbb{E}\big[X_t^{\beta,o}\big]d\beta + \int_\alpha^1 (1-\beta)\mathbb{E}\big[X_t^{\beta,o}\big]d\beta,$$

Differentiating with respect to the index α yields:

$$\frac{\partial}{\partial\alpha}Z_t^{\alpha,g} = -\int_0^\alpha \mathbb{E}\big[X_t^{\beta,o}\big]d\beta,\ \forall t \in [0,1]. \tag{17}$$

from which it follows that, whenever $\lambda = 0 \in [0,1]$:

$$\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\bigg|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T].$$

That is to say, for the uniform attachment graphon, the root node is a mean critical node.

These examples indicate that the structure of the networks modelled by these graphs play a key role in the interaction between agents in the associated GMFGs.

4. STATIONARITY PROPERTIES OF THE VALUE FUNCTIONS

In this section, we show that, under specific conditions, mean critical nodes can be readily identified as nodes at which the value functions are stationary. This result allows for the identification of mean critical nodes directly from the solutions to the Control Affine GMFG equations (13) and (14).

Proposition 3. Let Assumptions A1 and A2 hold, let $c(x) = 0, x \in \mathbb{R}$, and assume that the solution to the Control Affine GMFG problem admits nodes denoted $\lambda \in [0,1]$ at which the value function is stationary, that is

$$\frac{\partial V(\alpha,t,x)}{\partial\alpha}\bigg|_{\alpha=\lambda} = 0,\ \forall(t,x) \in [0,T]\times\mathbb{R}. \tag{18}$$

Then these nodes are mean critical nodes for the Control Affine GMFG system, that is to say at these nodes the local mean field is stationary:

$$\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\bigg|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T]. \tag{19}$$

Conversely, subject to the same conditions, mean field critical nodes are nodes at which the Control Affine GMFG value function is stationary.

Proof.

Differentiating the Control Affine GMFG equations (13)-(14) with respect to α yields the function

$$W(\alpha,t,x) := \frac{\partial V(\alpha,t,x)}{\partial\alpha},\ \forall(\alpha,t,x) \in [0,1]\times[0,T]\times\mathbb{R},$$

as a solution to the PDE:

$$\begin{aligned}
-\frac{\partial W(\alpha,t,x)}{\partial t} &= -q\gamma(t)\Big(x-\gamma(t)Z_t^{\alpha,g}\Big)^{-3} \\
&\times \exp\left[-\frac{q}{2}\Big(x-\gamma(t)Z_t^{\alpha,g}\Big)^{-2}\right]\frac{\partial}{\partial\alpha}\Big(Z_t^{\alpha,g}\Big) \\
&+ (a(x)+c(x)Z_t^{\alpha,g})\frac{\partial W(\alpha,t,x)}{\partial x} \\
&+ c(x)\frac{\partial}{\partial\alpha}\Big(Z_t^{\alpha,g}\Big)\frac{\partial V(\alpha,t,x)}{\partial x} \\
&- \frac{b^2(x)}{r}\frac{\partial V(\alpha,t,x)}{\partial x}\frac{\partial W(\alpha,t,x)}{\partial x} \\
&+ \frac{\sigma^2}{2}\frac{\partial^2 W(\alpha,t,x)}{\partial x^2}, \\
&\qquad (\alpha,t,x)\in[0,1]\times[0,T]\times\mathbb{R}
\end{aligned} \tag{20}$$

with terminal conditions

$$W(\alpha,T,x) = 0, \quad (\alpha,x)\in[0,1]\times\mathbb{R}.$$

Recalling that $c(x) = 0, x \in \mathbb{R}$, we see that at any given $\lambda \in [0,1]$ for which

$$W(\lambda,t,x) = 0, \qquad (t,x)\in[0,T]\times\mathbb{R},$$

the PDE (20) for $W(\cdot,\cdot,\cdot)$ takes the form

$$\begin{aligned}
0 &= -q\gamma(t)\Big(x-\gamma(t)Z_t^{\lambda,g}\Big)^{-3}\exp\left[-\frac{q}{2}\Big(x-\gamma(t)Z_t^{\lambda,g}\Big)^{-2}\right] \\
&\times\left(\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\bigg|_{\alpha=\lambda}\right), \qquad (t,x)\in[0,T]\times\mathbb{R},
\end{aligned} \tag{21}$$

and hence

$$\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\bigg|_{\alpha=\lambda} = 0, \quad t \in [0,T]. \tag{22}$$

Consequently $\lambda \in [0,1]$ is a mean field critical node.

The converse implication of the proposition holds since the boundary condition for the $W(\cdot,\cdot,\cdot)$ function is

$$W(\alpha,T,x) = \frac{\partial V(\alpha,T,x)}{\partial\alpha} = 0, \quad (\alpha,x)\in[0,1]\times\mathbb{R},$$

due to the boundary condition on the value function being $V(\alpha,T,x) = 0, (\alpha,t,x) \in [0,1]\times[0,T]\times\mathbb{R}$.

But then setting

$$\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\bigg|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T], \tag{23}$$

in the PDE (20) for $W(\cdot,\cdot,\cdot)$ results in the unique solution satisfying

$$\frac{\partial V(\alpha,t,x)}{\partial\alpha}\bigg|_{\alpha=\lambda} = W(\alpha,t,x)|_{\alpha=\lambda} = 0,\ \forall(t,x)\in[0,T]\times\mathbb{R}, \tag{24}$$

as required.

5. CONCLUSION

In this paper a class of Graphon Mean Field Games with control affine non-linear dynamics and exponentiated negative inverse quadratic (ENIQ) cost functions has been considered. Subject to the assumption of the existence and uniqueness of solutions to the relevant GMFG equations, it has been shown that a node at which the equilibrium Nash value is stationary with respect to location is such that the local mean field is also stationary with respect to location and conversely. In future work the analysis will be extended with proofs of the existence and uniqueness of all GMFG equations which arise in the current case and in the following extensions: (i) the class of systems where the dynamics of each agent are also an affine function of the local mean field, (ii) the consideration of different varieties of running costs, including quadratic and logistic, and (iii) the analysis of the influence on equilibria of specified classes of embedded graphon limits in arbitrary finite dimensions.

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