Accepted by the IEEE Conference on Decision and Control,
Rio de Janeiro, Brazil, Dec 2025

Perturbation Analysis with Q-Noise in LQG Graphon Mean Field Games

Tao Zhang, Peter E. Caines, Shuang Gao

***Abstract*— This paper presents a perturbation analysis for linear quadratic Gaussian graphon mean field games (LQG-GMFGs) with Q-noise. The perturbation response function is derived under both initial-state and Q-noise perturbations within a finite horizon framework. The theoretical results show that the alignment of eigenfunctions between the graphon in the dynamics and the perturbation graphon determines the bounds of the perturbation response function. Numerical simulations validate these findings, demonstrating the impact of eigenfunction alignment on the system's behavior.**

I. INTRODUCTION

A. Background

Mean field game (MFG) theory provides a mathematical framework for modeling and analyzing the behavior of large populations of interacting agents, where individual decisions are influenced by the collective state of the system [1], [2]. Originally developed to address stochastic dynamic games in large-scale systems, MFGs have found applications across diverse domains, including economics [3], autonomous vehicle coordination [4], and reinforcement learning [5], [6]. The introduction of network structures into MFG models has further enriched this framework, giving rise to graphon mean field games (GMFGs), which capture the effects of heterogeneous interactions among agents embedded in a network [2], [7]–[9]. In particular, the linear quadratic Gaussian (LQG) variant of GMFGs (LQG-GMFGs) offers a tractable setting to study the interplay between agent dynamics, network topology, and control strategies [8].

Real-world network systems are often subject to sources of uncertainty such as spatially correlated noise affecting the dynamics of all agents. Q-noise, as introduced into graphon mean field games and control in [10], [11], generalizes traditional Wiener processes by allowing spatial correlations defined by a covariance graphon, making it particularly suited to network systems where noise impacts are not independent across agents. While the sensitivity to initial state perturbations has recently been investigated for LQG-GMFGs, establishing a foundational framework for such analysis [12], the impact of Q-noise has not been considered in that context.

B. Contribution

In this paper, we integrate Q-noise into the LQG-GMFG framework and analyze the system's response to both local initial state perturbations and global noise disturbances. By employing a graphon spectral decomposition approach, explicit solutions are derived for the perturbation response function, capturing the combined effects of deterministic initial state shifts and stochastic Q-noise perturbations. The analysis introduces a perturbed Q-noise covariance structure to reflect practical scenarios, such as environmental fluctuations, and quantifies how these disturbances interact with the underlying network topology. The results give insight into the roles of network structure, noise correlation, and dynamic parameters in determining system perturbation.

C. Notation and Definitions

Throughout this paper, scalars and finite-dimensional vectors are denoted by standard non-bold letters (e.g., a, A). Infinite dimensional variables and operator kernels on Hilbert spaces are denoted by bold letters (e.g., $\mathbf{z}$, $\mathbf{M}$).

Let $\mathcal{X} \triangleq [0,T] \times [0,1] \times \Omega$, where Ω is a sample space associated with the Q-noise process. We equip $\mathcal{X}$ with the probability space $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P})$, where $\mathcal{B}(\mathcal{X})$ is the Borel σ-algebra on $[0,T] \times [0,1] \times \Omega$, and $\mathbb{P}$ is a probability measure. Consider a Hilbert space $L^2([0,T] \times [0,1] \times \Omega)$ equipped with the norm $\|\mathbf{z}\|_{L^2} \triangleq \left[\mathbb{E} \int_0^T \int_0^1 \mathbf{z}^2(t,\alpha) d\alpha dt\right]^{1/2} < \infty$ for any measurable random variable $\mathbf{z} \in L^2([0,T] \times [0,1] \times \Omega)$. Let $\mathcal{L}(L^2[0,1])$ denote the Banach algebra of bounded operators from $L^2[0,1]$ to itself, with $\mathbb{I}$ as the identity operator such that $[\mathbb{I}u](\alpha) = u(\alpha)$ and $[A\mathbb{I}u](\alpha) = Au(\alpha)$ for $u \in L^2[0,1]$, $A \in \mathbb{R}$. I_N is the $N \times N$ identity matrix and $\mathbf{1}(\alpha) = 1$ for any $\alpha \in [0,1]$.

Let $\{W_r(t)\}_{r=1}^K$ be a Brownian motion sequence over $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{P})$. Let $\mathcal{F}_t^0 \triangleq \sigma\left(\{W_r(s)\}_{r=1}^K; 0 \le s \le t\right)$ and $\mathcal{L}_{\mathcal{F}^0}(\Omega, [0,T]; \mathbb{R})$ denote the set of all $\{\mathcal{F}_t^0\}_{0 \le t \le T}$-adapted and measurable processes $X(t) \in \mathbb{R}$ such that $\mathbb{E}[\int_0^T X_t^2 dt] < \infty$. In what follows, we use $\mathcal{L}_{\mathcal{F}^0}$ to represent $\mathcal{L}_{\mathcal{F}^0}(\Omega, [0,T]; \mathbb{R})$ for simplicity. Let $\mathcal{H}$ be a Hilbert space. Denote by $\mathcal{M}^0(\mathcal{H})$ the space of $\mathcal{F}_T^0$-measurable random variables X such that $X(\omega) \in \mathcal{H}$ for $\omega \in \Omega$ almost surely.

Following [13], we denote by $\mathcal{L}_c^1$ the set of functions $f \in L^2([0,T] \times \Omega; \mathbb{R})$ such that

- $f \in \mathcal{L}_{\mathcal{F}^0}$ for almost every $\alpha \in [0,1]$,
- $f(\cdot, \omega)$ is continuous in $[0,T]$, almost surely $\omega \in \Omega$;

denote by $\mathcal{L}_L$ the set of functions $f \in L^2([0,T] \times [0,1] \times \Omega; \mathbb{R})$ such that

- $f(t, \cdot, \omega) \in L^1[0,1]$ for $t \in [0,T]$ and $\omega \in \Omega$ almost surely
- $f(\cdot, \alpha, \cdot) \in \mathcal{L}_{\mathcal{F}^0}$ for almost every $\alpha \in [0,1]$,

and denote by $\mathcal{L}_c \subset \mathcal{L}_L$ the set of functions $f \in \mathcal{L}_L$ such that $f(\cdot, \cdot, \omega)$ is continuous in $[0,T]$ and measurable in $[0,1]$.

T. Zhang and P. E. Caines are with the Department of Electrical and Computer Engineering, McGill University, Montréal, QC, H3A 0E9, Canada (e-mail: tao.zhang6@mail.mcgill.ca; peterc@cim.mcgill.ca). S. Gao is with the Department of Electrical Engineering, Polytechnique Montréal, Montréal, Canada (e-mail: shuang.gao@polymtl.ca). This work is supported by AFOSR Grant FA9550-23-1-0015, NSERC (Canada) Grant RGPIN-2019-05336 (PEC), and NSERC (Canada) Grant RGPIN-2024-06612 (SG).

II. Preliminaries

A. Graphs and Graphons

A finite undirected graph $G = (V, E)$ contains a vertex set $V = \{1, \dots, N\}$ and an edge set $E \subset V \times V$. Its weighted adjacency matrix $M^{[N]} = (M^{[N]}_{ij})$ satisfies $M^{[N]}_{ij} = 0$ if $(i, j) \notin E$ and nonzero otherwise. A graphon is a bounded symmetric Lebesgue measurable function $\mathbf{M} : [0, 1]^2 \to \mathbb{R}$. The cut metric topology [14] makes the space $\mathcal{W}$ of graphons compact and hence all graph sequences necessarily contain a convergent subsequence. Let a graph sequence $\{G_N\}_{N=1,2...}$ converge to a graphon $\mathbf{M}$ as the number of vertices $N \to \infty$. When $\mathbf{M} : [0, 1]^2 \to [0, 1]$, the element of $M^{[N]}$ and the corresponding limit $\mathbf{M}$ may be interpreted as the connection probabilities among nodes. The 2-norm of a graphon is defined as (see [14])

$$\|\mathbf{M}\|_2 \triangleq \left[\int_{[0,1]^2} \mathbf{M}(x, y)^2 dxdy\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{\infty} \lambda_i^2\right]^{\frac{1}{2}} \tag{1}$$

where $\{\lambda_i\}$ are eigenvalues of $\mathbf{M}$. The linear operator associated with $\mathbf{M}$ is defined as follows: for any $u \in L^2[0, 1]$,

$$[\mathbf{M}u](\cdot) \triangleq \int_{[0,1]} \mathbf{M}(\cdot, \beta)u(\beta)d\beta. \tag{2}$$

Let $\{P_i\}_{i=1}^N$ denote a finite uniform partition of $[0, 1]$, where $P_i \triangleq (\frac{i-1}{N}, \frac{i}{N}]$ for $i > 1$ and $P_1 \triangleq [0, \frac{1}{N}]$. For the adjacency matrix of any undirected graph denoted by M, we can identify a corresponding step-function graphon $\mathbf{M}^{[N]} \in \mathcal{W}$ such that

$$\mathbf{M}^{[N]}(\alpha, \beta) \triangleq M_{ij}, \quad \alpha \in P_i, \beta \in P_j; \tag{3}$$

similarly we can associate a piece-wise-constant function $\mathbf{z}^{[N]} \in L^2[0, 1]$ with a vector $z \in \mathbb{R}^N$ such that $\mathbf{z}^{[N]}(\alpha) \triangleq z^i, \alpha \in P_i$ (see [8]).

Definition 1 (Step-Wise Convergence of Graphons [9])**.** For the step-graphon sequence $\{\mathbf{M}^{[N]}\}$ on $\mathcal{W}$, we say that $\mathbf{M}^{[N]}$ converges step-wise to $\mathbf{M} \in \mathcal{W}$ if

$$\lim_{N\to\infty} \max_{1\le i\le N} \sum_{j=1}^{N} \left| \int_{P_j} \mathbf{M}^{[N]}(\alpha_i, \beta) - \mathbf{M}(\alpha_i, \beta)d\beta \right| = 0, \tag{4}$$

where α_i is the midpoint of $P_i = (\frac{i-1}{N}, \frac{i}{N}]$.

B. Q-Noise Processes

Define $\mathcal{Q}_1 \subset \mathcal{W}$ as the set of bounded, non-negative, symmetric functions $\mathbf{Q}$ with $\|\mathbf{Q}\|_2 = 1$ and hence, for any $f \in L^2[0, 1]$, such a $\mathbf{Q}$ satisfies

$$0 \le \int_0^1 \int_0^1 \mathbf{Q}(x, y)f(x)f(y)\, dx\, dy < \infty.$$

Definition 2 ([10], [11])**.** Q-noise processes are $L^2[0, 1]$-valued random processes that satisfy the following axioms.

1) Let $\mathbf{Q} \in \mathcal{Q}_1$, and let $w(\alpha, t, \omega) : \mathcal{X} \to \mathbb{R}$ be a measurable random variable over $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ for all $t \in [0, T], \alpha \in [0, 1], \omega \in \Omega$.
2) For all $\alpha \in [0, 1]$, $w(\alpha, t) - w(\alpha, s)$ is a Wiener process increment in time for all $t, s \in [0, T]$, with $w(\alpha, t) - w(\alpha, s) \sim \mathcal{N}(0, |t - s|\mathbf{Q}(\alpha, \alpha))$, where $w(\alpha, 0) = 0$ for all $\alpha \in [0, 1]$.
3) Let $w_{t-t'}(\alpha) = w(\alpha, t) - w(\alpha, t')$. Then,
$$\mathbb{E}[w_{t-t'}(\alpha)w_{s-s'}(\beta)] = |[t, t'] \cap [s, s']| \cdot \mathbf{Q}(\alpha, \beta). \tag{5}$$
4) For almost all $s, t \in [0, T], \alpha, \beta \in [0, 1]$, and $\omega \in \Omega$, $w(\alpha, t, \omega) - w(\beta, s, \omega)$ is Bochner-integrable as a function taking values in the Banach space of almost surely piece-wise continuous functions of $[s, t] \in [0, T]$.

For a covariance matrix $Q \in \mathbb{R}^{N\times N}$ with eigen-pairs $\{(\nu_r, h_r)\}_{r=1}^N$ with $\nu_r \ge 0$ (resp. $h_r \in \mathbb{R}^N$) denoting the eigenvalue (resp. eigenvector), let

$$v_t^i \triangleq \sum_{r=1}^{N} \sqrt{\nu_r} h_r^i W_r(t) \tag{6}$$

where h_r^i represents the i-th entry of h_r.

To ensure that the spatial correlation of the noise is consistently modeled in the transition from finite dimension to its mean-field limit, it is assumed that the step-graphon $\mathbf{Q}^{[N]}$ associated with Q converges step-wise to $\mathbf{Q}$, then the step-function $\mathbf{v}_t^{[N]}$ associated with $v_t \triangleq [v_t^1, \cdots, v_t^N]^\top$ converges to $\mathbf{v}_t \in L^2[0, 1]$, where

$$\mathbf{v}_t(\cdot) = \sum_{r=1}^{\infty} \sqrt{\nu_r}\psi_r(\cdot)W_r(t) \tag{7}$$

is a $\mathbf{Q}$-noise process defined over $\mathcal{X}$ with $\psi_r \in L^2[0, 1]$.

Assumption 1. The operators $\mathbf{Q}$ and $\mathbf{M}$ are assumed to be finite-rank in the sense that

$$\mathbf{Q}(\alpha, \beta) = \sum_{\ell=1}^{K} \nu_\ell \psi_\ell(\alpha)\psi_\ell(\beta)$$

$$\mathbf{M}(\alpha, \beta) = \sum_{\ell=1}^{d} \lambda_\ell \mathbf{f}_\ell(\alpha)\mathbf{f}_\ell(\beta),$$

for some integers $0 < K < \infty$ and $0 < d < \infty$, where $\{(\nu_\ell, \psi_\ell)\}_{\ell=1}^K$ and $\{(\lambda_\ell, \mathbf{f}_\ell)\}_{\ell=1}^d$ are the corresponding non-zero eigen-pairs.

Remark 1. The finite-rank assumption on $\mathbf{Q}$ and $\mathbf{M}$ is made for analytical tractability. This is because any general Hilbert-Schmidt operator $\mathbf{S} \in \mathcal{W}$ can be arbitrarily well-approximated in the L^2 norm by a finite-rank operator. Such an approximation is obtained by truncating its infinite spectral decomposition series after a sufficiently large number of terms (see [14, Chapter 7.5]).

C. Adapted Solutions to FBSDEs

We first present some generic results that support our main analysis. Consider the scalar forward-backward stochastic differential equation (FBSDE) with $0 < K < \infty$ independent Brownian motions:

$$\begin{cases} \begin{bmatrix} dz_t \\ ds_t \end{bmatrix} = A(t) \begin{bmatrix} z_t \\ s_t \end{bmatrix} dt + \sum_{k=1}^{K} \begin{bmatrix} \sigma^k \\ q_t^k \end{bmatrix} dW_k(t) \\ z_0 = \xi_0, \quad s_T = Qz_T \in \mathcal{M}^0(\mathbb{R}) \end{cases} \tag{8}$$

where $\sigma^k, Q \in \mathbb{R}$, and $A(t) \triangleq \begin{bmatrix} A_1(t) & B_1(t) \\ A_2(t) & B_2(t) \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is continuous in time.

Definition 3 (Adapted Solution [15])**.** An *adapted solution* to the FBSDE (8) is a triplet of processes $(z, s, \{q^k\}_{k=1}^K)$ defined on $[0,T]$ satisfying the following conditions:

1) z_t and s_t are continuous, $\{\mathcal{F}_t^0\}_{0\le t\le T}$-adapted processes such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |z_t|^2\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\sup_{0\le t\le T} |s_t|^2\right] < \infty.$$

2) For each $k \in \{1,\ldots,K\}$, q_t^k is an $\{\mathcal{F}_t^0\}_{0\le t\le T}$-adapted and measurable process such that $\mathbb{E}\left[\int_0^T |q_t^k|^2 dt\right] < \infty$.
3) The following hold almost surely for all $t \in [0,T]$:

$$\begin{cases} z_t = z_0 + \displaystyle\int_0^t (A_1(\tau)z_\tau + B_1(\tau)s_\tau)\, d\tau + \sum_{k=1}^K \sigma^k dW_k(\tau), \\ s_t = s_T - \displaystyle\int_t^T (A_2(\tau)z_\tau + B_2(\tau)s_\tau)\, d\tau - \sum_{k=1}^K q_\tau^k dW_k(\tau). \end{cases} \tag{9}$$

Lemma 1. *Let $(z, s, \{q^k\}_{k=1}^K)$ be $\{\mathcal{F}_t^0\}_{0\le t\le T}$-adapted and measurable processes over $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ taking values in $\mathbb{R}$, and let $\xi_0 \in \mathbb{R}$ be a deterministic value. If the following scalar Riccati ODE has a unique solution $P : [0,T] \to \mathbb{R}$:*

$$\begin{cases} -\dfrac{dP}{dt} = B_1(t)P^2 + (A_1(t) - B_2(t))P - A_2(t) \\ P(T) = Q, \end{cases} \tag{10}$$

then the FBSDE (8) *has a unique adapted solution $(z, s, \{q^k\}_{k=1}^K)$ where z_t is the unique solution to the forward SDE:*

$$dz_t = (A_1(t) + B_1(t)P(t))z_t dt + \sum_{k=1}^K \sigma^k dW_k(t), \quad z_0 = \xi_0, \tag{11}$$

$s_t = P(t)z_t$ and $q_t^k = P(t)\sigma^k$.

Proof. We first verify that the proposed $(z, s, \{q^k\}_{k=1}^K)$ to (11) is an adapted solution to (8). Let z_t be the solution of (11). We define $s_t = P(t)z_t$ and $q_t^k = P(t)\sigma^k$. The forward part of the FBSDE is satisfied by construction.

Apply Itô's product rule to $s_t = P(t)z_t$:

$$ds_t = \left[\dot{P}(t) + P(t)(A_1(t) + B_1(t)P(t))\right] z_t dt + \sum_{k=1}^K P(t)\sigma^k dW_k(t).$$

Clearly, (10) is specifically the condition that makes the drift term equal to $(A_2(t)z_t + B_2(t)s_t)dt$. The diffusion term is $\sum_{k=1}^K q_t^k dW_k(t)$ by definition. The boundary condition $s_T = P(T)z_T = Qz_T$ is also met. Thus, $(z, s, \{q^k\})$ is a solution.

We proceed to prove uniqueness. Let $(\tilde{z}, \tilde{s}, \{\tilde{q}^k\})$ be any adapted solution. Define the difference process $\Delta_t = \tilde{s}_t - P(t)\tilde{z}_t$ with $\Delta_T = Q\tilde{z}_T - Q\tilde{z}_T = 0$. Applying Itô's formula and (10) leads to:

$$d\Delta_t = (B_2(t) - P(t)B_1(t))\Delta_t dt + \sum_{k=1}^K (\tilde{q}_t^k - P(t)\sigma^k) dW_k(t).$$

One can easily verify that $\Delta_t = 0, \tilde{q}_t^k = P(t)\sigma^k$ is the trivial and unique solution to the above backward SDE for all $t \in [0,T]$ (see [15, Thm 4.2]). Furthermore, $\tilde{z}_t$ follows the same forward SDE as in (11). Hence, $z_t = \tilde{z}_t$ since (11) has unique solutions and $\tilde{s}_t = P(t)\tilde{z}_t = s_t$. Therefore, the solution is unique. □

D. Stochastic Games on Finite Networks with Q-Noise

Consider a network of N nodes, with each node associated with a cluster $\mathcal{C}_q$, $1 \le q \le N$, and of total population $L = \sum_{q=1}^N |\mathcal{C}_q|$, where $|\mathcal{C}_q|$ represents the number of agents within cluster $\mathcal{C}_q$. Let $\{\omega_t^i\}_{1\le i\le L}$ be standard i.i.d. Wiener processes independent of $\{W_r(s)\}_{r=1}^K$. Define the admissible control set $\mathcal{U}^{[N]}$ as the set of $\{\mathcal{F}_t^{[N]}\}_{0\le t\le T}$-adapted measurable processes u_t with $\mathbb{E}[\int_0^T |u_t|^2 dt] < \infty$ and

$$\mathcal{F}_t^{[N]} \triangleq \sigma\left(\{W_r(s)\}_{r=1}^K, \{\omega_s^i\}_{i=1}^L, \{x_0^i\}_{i=1}^L; 0 \le s \le t\right)$$

where x_0^i is the initial state of $\mathcal{A}_i$ and $\mathbb{E}[|x_0^i|^2] < \infty$.

For a generic agent $\mathcal{A}_i \in \mathcal{C}_q$ with $1 \le i \le L$ and $1 \le q \le N$, with the state x_t^i, control $u_t^i \in \mathcal{U}^{[N]}$, and network weighted empirical mean field $z_t^q \in \mathcal{L}_c^1$, the dynamics are given by

$$dx_t^i = (A_t x_t^i + B_t u_t^i + D_t z_t^q)dt + \sigma d\omega_t^i + \kappa dv_t^q \tag{12}$$

where $A_t, B_t, D_t : [0,T] \to \mathbb{R}$ are assumed to be continuous, $\sigma, \kappa > 0$, $v_t = [v_t^1, \cdots, v_t^N]^\top \in \mathbb{R}^N$ is the cluster-wise common noise vector introduced in (6) associated with $\{W_r(t)\}_{r=1}^K$, and the *network mean field* (perceived at cluster $\mathcal{C}_q$) is defined cluster-wise as

$$z_t^q \triangleq \frac{1}{N}\sum_{l=1}^N M_{ql}\mathbb{E}\left[\bar{x}_t^l | \mathcal{F}_t^0\right], \tag{13}$$

with

$$\bar{x}_t^q \triangleq \frac{1}{|\mathcal{C}_q|}\sum_{i=1}^{|\mathcal{C}_q|} x_t^i \tag{14}$$

Let the subpopulation of each cluster $\mathcal{C}_q$ go to infinity, i.e., $|\mathcal{C}_q| \to \infty$. Then $\bar{x}_t^q \to \int_{\mathbb{R}} x\mu_t^q(dx)$ by the Law of Large Numbers where μ_t^q denotes the state probability distribution function for cluster $\mathcal{C}_q$ at time t.

The cost functional of a generic agent $\mathcal{A}_i$ in cluster $\mathcal{C}_q$ is defined as

$$J^i(u_i) \triangleq \mathbb{E}\Big[\int_0^T (\|x_t^i - y_t^q\|_{Q_t}^2 + \|u_t^i\|_{R_t}^2)dt + \|x_T^i - y_T^q\|_{Q_T}^2\Big] \tag{15}$$

where $y_t^q \triangleq Hz_t^q + \eta$, $Q_t, Q_T \ge 0, R_t > 0$ for $t \in [0,T]$, $H, \eta \in \mathbb{R}$.

Let $\bar{x}_t \triangleq \left[\bar{x}_t^1, \cdots, \bar{x}_t^N\right]^\top, z_t \triangleq \left[z_t^1, \cdots, z_t^N\right]^\top, s_t \triangleq \left[s_t^1, \cdots, s_t^N\right]^\top \in \mathbb{R}^N$. Then taking averages over the nodal populations of both sides of (12) as $|\mathcal{C}_q| \to \infty$ for all $1 \le q \le N$, and assuming the limits are well-defined and exist, we obtain

$$d\bar{x}_t = (A_t I_N \bar{x}_t + B_t I_N \bar{u}_t + D_t I_N z_t)dt + \kappa dv_t \qquad (16)$$

where $\bar{u}_t = [\bar{u}_t^1, \cdots, \bar{u}_t^N]^\top$ with components defined by

$$\bar{u}_t^q = \lim_{|\mathcal{C}_q|\to\infty} \frac{1}{|\mathcal{C}_q|} \sum_{i=1}^{|\mathcal{C}_q|} u_t^i.$$

Remark 2. The idiosyncratic noise ω^i disappears due to the Law of Large Numbers, given that the individual Wiener processes $\{\omega^i\}$ are assumed to be i.i.d. for each agent.

Remark 3. The global perturbation v_t characterized by (7) reflects the inherent influence of stochastic noise on every node. This noise exhibits spatial correlation across nodes, capturing the interconnected nature of the network.

Definition 4 (Best Response and Nash Equilibrium in Finite Networks with Finite Populations)**.** For an agent $\mathcal{A}_i$, given the controls $u^{-i} \triangleq (u^1, \cdots, u^{i-1}, u^{i+1}, \cdots, u^L)$ of all other agents, the *best response* of agent $\mathcal{A}_i$ is defined by $\arg\inf_{u\in\mathcal{U}^{[N]}} J^i(u, u^{-i})$.

A control tuple $(u^1, \cdots, u^L) \in \mathcal{U}^{[N]} \times \cdots \times \mathcal{U}^{[N]}$ is said to be a *Nash equilibrium* if

$$J^i(u^i, u^{-i}) \le \inf_{v\in\mathcal{U}^{[N]}} J^i(v, u^{-i}),\ \forall i = 1, 2, \cdots, L$$

that is, for all agents $\mathcal{A}_i$, u^i is a best response to u^{-i} [16].

III. Perturbation Analysis with Q-Noise

A. Initial State Perturbation on Finite Networks

In addition to the global Q-noise perturbation v_t in (12), we further consider a local initial state perturbation

$$x_0^{(\epsilon,q),i} \triangleq x_0^i + \epsilon \in \mathbb{R} \qquad (17)$$

of a generic agent $\mathcal{A}_i$ in cluster $\mathcal{C}_q, 1 \le q \le N$ where $\epsilon > 0$, that is, the same ϵ perturbation on initial conditions applies to all agents in the cluster $\mathcal{C}_q$, then the averaged initial state vector is

$$\bar{x}_0^{(\epsilon,q)} \triangleq \bar{x}_0 + \epsilon \mathbf{1}^q \in \mathbb{R}^N \qquad (18)$$

where $\mathbf{1}^q \in \mathbb{R}^N$ is an all-zero vector except at index q. Let $x_t^{(\epsilon,q),i}, \bar{x}_t^{(\epsilon,q)}, z_t^{(\epsilon,q)}, s_t^{(\epsilon,q)}$ denote the perturbed variables from $x_t^i, \bar{x}_t, z_t, s_t$. The perturbed dynamics of $\mathcal{A}_i$ are given by

$$dx_t^{(\epsilon,q),i} = (A_t x_t^{(\epsilon,q),i} + B_t u_t^{(\epsilon,q),i} + D_t z_t^{(\epsilon,q)})dt + \sigma d\omega_t^i + \kappa\epsilon dv_t^q \qquad (19)$$

and the averaged state vector $\bar{x}_t^{(\epsilon,q)} \in \mathbb{R}^N$ with perturbed dynamics from (16) are

$$d\bar{x}_t^{(\epsilon,q)} = (A_t I_N \bar{x}_t^{(\epsilon,q)} + B_t I_N \bar{u}_t^{(\epsilon,q)} + D_t I_N \bar{z}_t^{(\epsilon,q)})dt + \kappa\epsilon dv_t \qquad (20)$$

with averaged cluster initial condition $\bar{x}_0^{(\epsilon,q)} \in \mathbb{R}^N$.

B. Initial State Perturbation in LQG-GMFGs

In the graph limit problem, i.e., $N \to \infty$, a representative agent is no longer identified by a discrete index q, but by a continuous label $\alpha \in [0, 1]$ if $\alpha \in P_q = (\frac{q-1}{N}, \frac{q}{N}]$.

Assumption 2. The graphon $\mathbf{M}$ is continuous in its arguments. In addition, $\mathbf{M}^{[N]}$ converges step-wise to $\mathbf{M}$.

We now follow the GMFG procedure [9] to construct the system dynamics of infinite populations on a graph sequence limit, namely, on a graphon. Let the number of clusters $N \to \infty$ and let Assumption 2 hold. These assumptions are needed for the so-called ε-Nash properties of the GMFG solutions (see [9], [13]).

For a generic agent $\mathcal{A}_\alpha$, define the admissible control set $\mathcal{U}^\alpha$ as the set of $\{\mathcal{F}_t^\alpha\}_{0\le t\le T}$-adapted measurable processes u_t with $\mathbb{E}[\int_0^T |u_t|^2 dt] < \infty$ and

$$\mathcal{F}_t^\alpha \triangleq \sigma\left(\{W_r(s)\}_{r=1}^K, \omega_s^\alpha, x_0^\alpha; 0 \le s \le t\right) \qquad (21)$$

where x_0^α is the initial state of $\mathcal{A}_\alpha$ and $\mathbb{E}[|x_0^\alpha|^2] < \infty$.

In the limit problem (with both the local population limits and the graph limit), consider a local initial state perturbation

$$\bar{\mathbf{x}}_0^{(\epsilon,\theta)} \triangleq \bar{\mathbf{x}}_0 + \epsilon\delta_\theta \qquad (22)$$

at a specific network location $\theta \in [0, 1]$ and a global $\mathbf{Q}$-noise perturbation where δ_θ is the Delta function centered at θ. Denote the perturbed variables from $\bar{\mathbf{x}}_t, \mathbf{z}_t, \mathbf{s}_t$ by $\bar{\mathbf{x}}_t^{(\epsilon,\theta)}, \mathbf{z}_t^{(\epsilon,\theta)}, \mathbf{s}_t^{(\epsilon,\theta)}$ where the perturbed graphon mean field $\mathbf{z}_t^{(\epsilon,\theta)}$ is given by

$$\mathbf{z}_t^{(\epsilon,\theta)} = \int_0^1 \mathbf{M}(\cdot, \beta)\mathbb{E}[\bar{\mathbf{x}}_t^{(\epsilon,\theta)}(\beta)|\mathcal{F}_t^0]d\beta. \qquad (23)$$

Define the infinite dimensional stochastic processes:

$$\tilde{\mathbf{x}}_t^{(\epsilon,\theta)} \triangleq \bar{\mathbf{x}}_t^{(\epsilon,\theta)} - \bar{\mathbf{x}}_t,\ \tilde{\mathbf{z}}_t^{(\epsilon,\theta)} \triangleq \mathbf{z}_t^{(\epsilon,\theta)} - \mathbf{z}_t,\ \tilde{\mathbf{s}}_t^{(\epsilon,\theta)} \triangleq \mathbf{s}_t^{(\epsilon,\theta)} - \mathbf{s}_t \qquad (24)$$

with initial conditions $\tilde{\mathbf{x}}_0^{(\epsilon,\theta)} = \epsilon\delta_\theta$.

Remark 4. The presence of δ_θ makes $\bar{\mathbf{x}}_0^{(\epsilon,\theta)}$ into a non-$L^2[0, 1]$ function, even if $\bar{\mathbf{x}}(0) \in L^2[0, 1]$. However, this does not affect the subsequent analysis, as we only require $\tilde{\mathbf{z}}_t^{(\epsilon,\theta)} \in L^2[0, 1]$ for any $t \in [0, T]$ in (34).

The dynamics of a representative agent $\mathcal{A}_\alpha$ in cluster $\mathcal{C}_\alpha$ with $\alpha \in [0, 1]$ are given by

$$\begin{aligned} d\mathbf{x}_t^{(\epsilon,\theta)}(\alpha) = (A_t\mathbf{x}_t^{(\epsilon,\theta)}(\alpha) + B_t\mathbf{u}_t^{(\epsilon,\theta)}(\alpha) + D_t\mathbf{z}_t^{(\epsilon,\theta)}(\alpha))dt \\ + \sigma d\omega_t(\alpha) + \kappa\epsilon d\mathbf{v}_t(\alpha) \end{aligned} \qquad (25)$$

where $\mathbf{u}_t^{(\epsilon,\theta)}(\alpha) \in \mathcal{U}^\alpha$, $\omega_t(\alpha)$ is a standard Brownian motion and $\mathbf{v}_t$ is a $\mathbf{Q}$-noise process. Agent $\mathcal{A}_\alpha$ aims to minimize the cost functional

$$\begin{aligned} J^\alpha(\mathbf{u}_t^{(\epsilon,\theta)}(\alpha)) \triangleq \mathbb{E}\Big[\int_0^T (\|\mathbf{x}_t^{(\epsilon,\theta)}(\alpha) - \mathbf{y}_t^{(\epsilon,\theta)}(\alpha)\|_{Q_t}^2 \\ + \|\mathbf{u}_t^{(\epsilon,\theta)}(\alpha)\|_{R_t}^2)dt + \|\mathbf{x}_T^{(\epsilon,\theta)}(\alpha) - \mathbf{y}_T^{(\epsilon,\theta)}(\alpha)\|_{Q_T}^2\Big] \end{aligned} \qquad (26)$$

where $\mathbf{y}_t^{(\epsilon,\theta)}(\alpha) \triangleq H\mathbf{z}_t^{(\epsilon,\theta)}(\alpha) + \eta$.

Definition 5 (Best Response and Nash Equilibrium in Infinite Networks with Infinite Populations)**.** For an agent $\mathcal{A}_\alpha$, given the controls $u^{-\alpha} \triangleq \bigcup_{\beta\in[0,1]/\{\alpha\}} u^\beta$ of all other agents, the *best response* of agent $\mathcal{A}_\alpha$ is defined by $\arg\inf_{u\in\mathcal{U}^\alpha} J^\alpha(u, u^{-\alpha})$.

A control tuple $(u^\alpha)_{\alpha\in[0,1]} \in \bigcup_{\alpha\in[0,1]} \mathcal{U}^\alpha$ is said to be a *Nash equilibrium* if

$$J^\alpha(u^\alpha, u^{-\alpha}) \le \inf_{v\in\mathcal{U}^\alpha} J^\alpha(v, u^{-\alpha}), \forall \alpha \in [0,1],$$

that is, for all agents $\mathcal{A}_\alpha$, u^α is a best response to $u^{-\alpha}$.

Define the transformed noise term $\mathbf{n}_t(\cdot) \triangleq \mathbf{M}\mathbf{v}_t(\cdot)$. Recall that $\mathbf{v}_t$ is a $\mathbf{Q}$-noise process and $\{(\nu_r,\psi_r)\}_{r=1}^K$are eigen pairs associated with $\mathbf{Q}$. Then $\mathbf{n}_t$ can be written as

$$\mathbf{n}_t(\cdot) = \sum_{r=1}^{K} \sqrt{\nu_r}[\mathbf{M}\psi_r](\cdot)W_r(t). \tag{27}$$

Let $A_c(t) \triangleq A_t - R_t^{-1}B_t^2\Pi_t \in \mathbb{R}$ with $(\Pi_t)_{t\in[0,T]}$ determined by the scalar Riccati equation

$$-\dot{\Pi}_t = 2A_t\Pi_t - B_t^2\Pi_t^2R_t^{-1} + Q_t, \quad \Pi_T = Q_T \in \mathbb{R}.$$

Define $\mathbb{A}(t) \triangleq [A_c(t)\mathbb{I}] \in \mathcal{L}(L^2[0,1])$.

Lemma 2. *Given a process $\mathbf{z}^{(\epsilon,\theta)} \in \mathcal{L}_c$, if there exists $(\mathbf{s}^{(\epsilon,\theta)}, \{\mathbf{q}^k\}_{k=1}^K) \in \mathcal{L}_c \times (\mathcal{L}_L)^K$ that satisfies the following backward SDE (BSDE):*

$$\begin{aligned} d\mathbf{s}_t^{(\epsilon,\theta)} = \Big(&-[\mathbb{A}(t)]\mathbf{s}_t^{(\epsilon,\theta)} + [(Q_tH - \Pi_tD_t)\mathbb{I}]\mathbf{z}_t^{(\epsilon,\theta)} \\ &+[Q_t\mathbb{I}]\eta\mathbf{1}\Big)\,dt + \sum_{k=1}^K q_t^k dW_k(t), \end{aligned} \tag{28}$$

then the optimal control is given by

$$\mathbf{u}_t^{(\epsilon,\theta)}(\alpha) = -R_t^{-1}B_t(\Pi_t\mathbf{x}_t^{(\epsilon,\theta)}(\alpha) + \mathbf{s}_t^{(\epsilon,\theta)}(\alpha)) \tag{29}$$

where $\mathbf{x}_t^{(\epsilon,\theta)}(\alpha)$ satisfies (25)*.*

Moreover, let $\mathbb{A}'(t) \triangleq \mathbb{A}(t) + D_t\mathbf{M} \in \mathcal{L}(L^2[0,1])$, then the LQG-GMFG has a Nash equilibrium if and only if $\mathbf{z}^{(\epsilon,\theta)}$ satisfies

$$\begin{aligned} \mathbf{z}_t^{(\epsilon,\theta)} = \mathbf{z}_0^{(\epsilon,\theta)} &+ \int_0^t \left([\mathbb{A}'(\tau)]\mathbf{z}_\tau^{(\epsilon,\theta)} - [\frac{B_\tau^2}{R_\tau}\mathbf{M}]\mathbf{s}_\tau^{(\epsilon,\theta)}\right)d\tau \\ &+ \int_0^t \kappa\epsilon\; d\mathbf{n}_\tau \end{aligned} \tag{30}$$

where $\mathbf{z}_0^{(\epsilon,\theta)} = [\mathbf{M}\bar{\mathbf{x}}_0^{(\epsilon,\theta)}]$.

Proof. The proof is a direct extension of the methodology in [13] for the single common noise case. This logic extends directly to the Q-noise setting, as the Q-noise process is assumed to be a finite linear combination of independent Wiener processes hence preserving the proof in [13]. ☐

Theorem 1 (Infinite Network FBSDE with Q-Noise)**.** *Assume that there exists a unique solution $(\mathbf{z}^{(\epsilon,\theta)}, \mathbf{s}^{(\epsilon,\theta)}, \{\mathbf{q}^k\}_{k=1}^K) \in \mathcal{L}_c \times \mathcal{L}_c \times (\mathcal{L}_L)^K$ for any $t\in[0,T]$ that are given by*

$$\begin{aligned} \mathbf{z}_t^{(\epsilon,\theta)} &= \mathbf{z}_0^{(\epsilon,\theta)} + \int_0^t \left([\mathbb{A}'(\tau)]\mathbf{z}_\tau^{(\epsilon,\theta)} - [\frac{B_\tau^2}{R_\tau}\mathbf{M}]\mathbf{s}_\tau^{(\epsilon,\theta)}\right)d\tau \\ &\quad + \int_0^t \kappa\epsilon\; d\mathbf{n}_\tau, \\ \mathbf{z}_0^{(\epsilon,\theta)} &= \int_{[0,1]} \mathbf{M}(\cdot,\beta)\bar{\mathbf{x}}_0^{(\epsilon,\theta)}(\beta)d\beta, \\ \mathbf{s}_t^{(\epsilon,\theta)} &= \mathbf{s}_T^{(\epsilon,\theta)} + \int_t^T \Big([\mathbb{A}'(\tau)]\mathbf{s}_\tau^{(\epsilon,\theta)} - [(Q_\tau H - \Pi_\tau D_\tau)\mathbb{I}]\mathbf{z}_\tau^{(\epsilon,\theta)} \\ &\quad -[Q_t\mathbb{I}]\eta\mathbf{1}\Big)\,dt - \sum_{k=1}^K \int_t^T \mathbf{q}_\tau^k dW_k(\tau), \\ \mathbf{s}_T^{(\epsilon,\theta)} &= [Q_T\mathbb{I}](H\mathbb{I}\mathbf{z}_T^{(\epsilon,\theta)} + \eta\mathbf{1}) \in \mathcal{M}^0(L^2[0,1]), \end{aligned} \tag{31}$$

and $(\mathbf{s}^{(\epsilon,\theta)}, \{\mathbf{q}^k\}_{k=1}^K)$ satisfies (28)*, then the limit game problem defined by* (25) *and* (26) *has a Nash equilibrium and the best response in the equilibrium is given as follows: for any agent $\mathcal{A}_\alpha \in \mathcal{C}_\alpha$ for almost every $\alpha\in[0,1]$ is given by*

$$\mathbf{u}_t^{(\epsilon,\theta)}(\alpha) = -R_t^{-1}B_t(\Pi_t\mathbf{x}_t^{(\epsilon,\theta)}(\alpha) + \mathbf{s}_t^{(\epsilon,\theta)}(\alpha)). \tag{32}$$

Proof. Given $\mathbf{z}^{(\epsilon,\theta)}$, if $(\mathbf{s}^{(\epsilon,\theta)}, \{\mathbf{q}^k\}_{k=1}^K)$ satisfies (28), then it follows from Lemma 2 that the best response is given by (29). In addition, since $(\mathbf{z}^{(\epsilon,\theta)}, \mathbf{s}^{(\epsilon,\theta)}, \{\mathbf{q}^k\}_{k=1}^K)$ satisfies (31), the LQG-GMFG has a Nash equilibrium by Lemma 2. ☐

The limiting dynamics in (20) under the best response in (29) are

$$\begin{aligned} d\bar{\mathbf{x}}_t^{(\epsilon,\theta)} = \Big([A_t\mathbb{I}]\bar{\mathbf{x}}_t^{(\epsilon,\theta)} - [\frac{B_t^2}{R_t}\mathbb{I}]\mathbf{s}_t^{(\epsilon,\theta)} + [D_t\mathbb{I}]\mathbf{z}_t^{(\epsilon,\theta)}\Big)\,dt \\ + \kappa\epsilon d\mathbf{v}_t. \end{aligned} \tag{33}$$

We derive the infinite dimensional perturbation FBSDE from (31) and (24):

$$\begin{aligned} d\tilde{\mathbf{z}}_t^{(\epsilon,\theta)} &= ([\mathbb{A}(t) + D_t\mathbf{M}]\tilde{\mathbf{z}}_t^{(\epsilon,\theta)} - [\frac{B_t^2}{R_t}\mathbf{M}]\tilde{\mathbf{s}}_t^{(\epsilon,\theta)})dt + \kappa\epsilon d\mathbf{n}_t, \\ \tilde{\mathbf{z}}_0^{(\epsilon,\theta)} &= \int_0^1 \mathbf{M}(\theta,\beta)\tilde{\mathbf{x}}_0^{(\epsilon,\theta)}(\beta)d\beta = \mathbf{M}(\theta,\cdot)\epsilon, \\ d\tilde{\mathbf{s}}_t^{(\epsilon,\theta)} &= \Big(-[\mathbb{A}(t)]\tilde{\mathbf{s}}_t^{(\epsilon,\theta)} + [(Q_tH - \Pi_tD_t)\mathbb{I}]\tilde{\mathbf{z}}_t^{(\epsilon,\theta)}\Big)\,dt \\ &\quad + \sum_{k=1}^K \mathbf{q}_t^k dW_k(t), \\ \tilde{\mathbf{s}}_T^{(\epsilon,\theta)} &= [Q_TH\mathbb{I}]\tilde{\mathbf{z}}_T^{(\epsilon,\theta)} \in \mathcal{M}^0(L^2[0,1]). \end{aligned} \tag{34}$$

C. Spectral Decomposition of the Perturbation FBSDE

Consider the orthonormal eigenfunctions $\{\mathbf{f}_\ell \in L^2[0,1]\}_{\ell=1}^\infty$ associated with non-zero eigenvalues $\{\lambda_\ell\}_{\ell=1}^\infty$ of $\mathbf{M}$ such that $[\mathbf{M}\mathbf{f}_\ell] = \lambda_\ell\mathbf{f}_\ell$ and $\|\mathbf{f}_\ell\|_{L^2} = 1$ for any $\alpha\in[0,1]$.[1] Let $\mathcal{S} \triangleq \text{span}(\{\mathbf{f}_\ell\}) \subset L^2[0,1]$ and denote its orthogonal complement by $\mathcal{S}^\perp$.

[1] $\mathbf{M}$ is a compact operator and has a spectral decomposition $\mathbf{M}(x,y) = \sum_{\ell=1}^\infty \lambda_\ell\mathbf{f}_\ell(\alpha)\mathbf{f}_\ell(\beta)$ for all $x,y\in[0,1]$, see [14, Chapter 7.5].

Uniqueness of the solution to (36) is guaranteed by the following monotonicity assumption.

Assumption 3 ([13], [17])**.** For all $x, y \in \mathbb{R}$, there exist constants $\beta_1^\ell > 0, \mu_1^\ell > 0$ (or $\beta_1^\ell < 0, \mu_1^\ell < 0$) such that

$$\begin{cases} D_t\lambda_\ell xy + (2Q_tH - D_t\Pi_t)\, x^2 - \frac{B_t^2}{2R_t}\lambda_\ell y^2 \le -\beta_1^\ell x^2, \\ Q_T H \le -\mu_1^\ell. \end{cases} \tag{35}$$

Theorem 2. *Let Assumptions 1 and 3 be in force. Then for any* $1 \le \ell \le d$, *there exists a unique adapted solution* $(\tilde{z}^{(\epsilon,\theta),\ell}, \tilde{s}^{(\epsilon,\theta),\ell}, \{q^{\ell,k}\}_{k=1}^K) \in \mathcal{L}_{\mathcal{F}^0} \times \mathcal{L}_{\mathcal{F}^0} \times (\mathcal{L}_{\mathcal{F}^0})^K$ *to the following scalar FBSDE system:*

$$\begin{cases} d\tilde{z}_t^{(\epsilon,\theta),\ell} = \left[(A_c(t) + \lambda_\ell D_t)\tilde{z}_t^{(\epsilon,\theta),\ell} - \lambda_\ell \dfrac{B_t^2}{R_t}\tilde{s}_t^{(\epsilon,\theta),\ell}\right] dt \\ \qquad\qquad + \kappa\epsilon\lambda_\ell \displaystyle\sum_{k=1}^K \sigma^{\ell,k} dW_k(t) \\ d\tilde{s}_t^{(\epsilon,\theta),\ell} = \left[-(Q_tH - \Pi_t D_t)\tilde{z}_t^{(\epsilon,\theta),\ell} - A_c(t)\tilde{s}_t^{(\epsilon,\theta),\ell}\right] dt \\ \qquad\qquad + \displaystyle\sum_{k=1}^K q_t^{\ell,k} dW_k(t) \\ \tilde{z}_0^{(\epsilon,\theta),\ell} = \lambda_\ell \mathbf{f}_\ell(\theta)\epsilon, \quad \tilde{s}_T^{(\epsilon,\theta),\ell} = Q_T H \tilde{z}_T^{(\epsilon,\theta),\ell} \in \mathcal{M}^0(\mathbb{R}), \end{cases} \tag{36}$$

where $\sigma^{\ell,k} \triangleq \sqrt{\nu_k}\langle \psi_k, \mathbf{f}_\ell\rangle$. *Furthermore, the solution to* (34) *is given by:*

$$\tilde{\mathbf{z}}^{(\epsilon,\theta)} = \sum_{\ell=1}^d \mathbf{f}_\ell \tilde{z}_t^{(\epsilon,\theta),\ell}, \ \ \tilde{\mathbf{s}}^{(\epsilon,\theta)} = \sum_{\ell=1}^d \mathbf{f}_\ell \tilde{s}_t^{(\epsilon,\theta),\ell}, \ \ \mathbf{q}^k = \sum_{\ell=1}^d \mathbf{f}_\ell q^{\ell,k} \tag{37}$$

and therefore the LQG-GMFG problem has a Nash equilibrium.

Proof. It follows from [8], [13] that $\tilde{\mathbf{z}}_t^{(\epsilon,\theta)}$ can be decomposed as

$$\tilde{\mathbf{z}}_t^{(\epsilon,\theta)} = \sum_{\ell=1}^d \mathbf{f}_\ell \langle \tilde{\mathbf{z}}_t^{(\epsilon,\theta)}, \mathbf{f}_\ell\rangle + \breve{z}_t^{(\epsilon,\theta)}(\mathbf{1} - \sum_{\ell=1}^d \mathbf{f}_\ell\langle \mathbf{f}_\ell, \mathbf{1}\rangle)$$

where $\breve{z}_t^{(\epsilon,\theta)} = 0$ for the whole time interval since $\tilde{\mathbf{z}}_t^{(\epsilon,\theta)} = \mathbf{M}\tilde{\mathbf{x}}_t^{(\epsilon,\theta)} \in \mathcal{S}$. Projecting $\tilde{\mathbf{z}}_t^{(\epsilon,\theta)}$ onto the eigen space associated with $\mathbf{f}_\ell$ yields

$$\begin{aligned} \tilde{z}_t^{(\epsilon,\theta),\ell} &\triangleq \langle \tilde{\mathbf{z}}_t^{(\epsilon,\theta)}, \mathbf{f}_\ell\rangle \\ &= \int_0^1\int_0^1 \sum_{k=1}^K \lambda_k \mathbf{f}_k(\alpha)\mathbf{f}_k(\beta)\tilde{\mathbf{x}}_t^{(\epsilon,\theta)}(\beta) d\beta \mathbf{f}_\ell(\alpha) d\alpha \\ &= \sum_{k=1}^K \lambda_k \int_0^1 \mathbf{f}_k(\alpha)\mathbf{f}_\ell(\alpha) d\alpha \int_0^1 \mathbf{f}_k(\beta)\tilde{\mathbf{x}}_t^{(\epsilon,\theta)}(\beta) d\beta \\ &= \lambda_\ell \langle \mathbf{f}_\ell, \tilde{\mathbf{x}}_t^{(\epsilon,\theta)}\rangle, \end{aligned} \tag{38}$$

then for $\alpha \in [0,1]$, $\tilde{\mathbf{z}}_t^{(\epsilon,\theta)}$ is given by

$$\begin{aligned} \tilde{\mathbf{z}}_t^{(\epsilon,\theta)}(\alpha) &= \int_0^1 \sum_\ell \lambda_\ell \mathbf{f}_\ell(\alpha)\mathbf{f}_\ell(\beta)\tilde{\mathbf{x}}_t^{(\epsilon,\theta)}(\beta) d\beta \\ &= \sum_\ell \mathbf{f}_\ell(\alpha)\lambda_\ell\langle \mathbf{f}_\ell, \tilde{\mathbf{x}}_t^{(\epsilon,\theta)}\rangle \\ &= \sum_\ell \mathbf{f}_\ell(\alpha)\tilde{z}_t^{(\epsilon,\theta),\ell}. \end{aligned} \tag{39}$$

The dynamics of ℓ-component are given by

$$\begin{aligned} d\tilde{z}_t^{(\epsilon,\theta),\ell} = {} & [(A_c(t) + \lambda_\ell D_t)\tilde{z}_t^{(\epsilon,\theta),\ell} - \lambda_\ell \frac{B_t^2}{R_t}\tilde{s}_t^{(\epsilon,\theta),\ell}]dt \\ & + \kappa\epsilon\langle d\mathbf{n}_t, \mathbf{f}_\ell\rangle \end{aligned}$$

with

$$\tilde{z}_0^{(\epsilon,\theta),\ell} = \lambda_\ell \int_0^1 \delta_\theta(\beta)\mathbf{f}_\ell(\beta)\epsilon d\beta = \lambda_\ell \mathbf{f}_\ell(\theta)\epsilon. \tag{40}$$

The noise term can be decomposed as $\mathbf{n}_t = \sum_{\ell=1}^d \mathbf{f}_\ell\langle \mathbf{n}_t, \mathbf{f}_\ell\rangle + \breve{\mathbf{n}}_t$ where

$$\begin{aligned} n_t^\ell \triangleq \langle \mathbf{n}_t, \mathbf{f}_\ell\rangle &= \langle [\mathbf{M}\mathbf{v}_t], \mathbf{f}_\ell\rangle = \lambda_\ell\langle \mathbf{v}_t, \mathbf{f}_\ell\rangle \\ &= \lambda_\ell \sum_{k=1}^K \sqrt{\nu_k}\langle \psi_k, \mathbf{f}_\ell\rangle W_k(t) \end{aligned} \tag{41}$$

with covariance

$$\begin{aligned} \mathbb{E}[n_t^\ell n_s^\ell] &= \int\int \mathbb{E}[\mathbf{n}_t(\alpha)\mathbf{n}_s(\beta)]\mathbf{f}_\ell(\alpha)\mathbf{f}_\ell(\beta) d\alpha d\beta \\ &= (t\wedge s)\int\int \mathbf{Q}(\alpha,\beta)\mathbf{f}_\ell(\alpha)\mathbf{f}_\ell(\beta) d\alpha d\beta \\ &= (t\wedge s)\lambda_\ell^2 \sum_{k=1}^K \nu_k |\langle \psi_k, \mathbf{f}_\ell\rangle|^2. \end{aligned}$$

Thus, we have

$$\begin{aligned} d\tilde{z}_t^{(\epsilon,\theta),\ell} = {} & [(A_c(t) + \lambda_\ell D_t)\tilde{z}_t^{(\epsilon,\theta),\ell} - \lambda_\ell \frac{B_t^2}{R_t}\tilde{s}_t^{(\epsilon,\theta),\ell}]dt \\ & + \kappa\epsilon\lambda_\ell \sum_{k=1}^K \sqrt{\nu_k}\langle \psi_k, \mathbf{f}_\ell\rangle dW_k(t). \end{aligned} \tag{42}$$

Similarly,

$$\tilde{\mathbf{s}}_t^{(\epsilon,\theta)} = \sum_{\ell=1}^d \mathbf{f}_\ell \tilde{s}_t^{(\epsilon,\theta),\ell} + \breve{s}_t^{(\epsilon,\theta)}(\mathbf{1} - \sum_{\ell=1}^d \mathbf{f}_\ell\langle \mathbf{f}_\ell, \mathbf{1}\rangle) \tag{43}$$

where $(\breve{s}_t^{(\epsilon,\theta)}, \{\breve{q}_t^k\})$ is the solution to

$$d\breve{s}_t^{(\epsilon,\theta)} = -A_c(t)\breve{s}_t^{(\epsilon,\theta)} dt + \sum_{k=1}^K \breve{q}_t^k dW_t^k, \quad \breve{s}_T^{(\epsilon,\theta)} = 0 \tag{44}$$

which has a unique solution $\breve{s}_t^{(\epsilon,\theta)} = 0, \breve{q}_t^k = 0$ for all $t \in [0,T]$, and there exists $\{q^{\ell,k}\}_{k=1}^K \in (\mathcal{L}_{\mathcal{F}^0})^K$ such that

$$\begin{aligned} \tilde{s}_t^{(\epsilon,\theta),\ell} = {} & \int_t^T \left(A_c(\tau)\tilde{s}_\tau^{(\epsilon,\theta),\ell} - (Q_\tau H - \Pi_\tau D_\tau)\,\tilde{z}_\tau^{(\epsilon,\theta),\ell}\right) d\tau \\ & - \sum_{k=1}^K \int_t^T q_t^{\ell,k} dW_k(\tau) + \tilde{s}_T^{(\epsilon,\theta),\ell}. \end{aligned}$$

For a fixed ℓ, we can identify its components with the notation used in the Lemma 1:

$$A_1^\ell(t) \triangleq A_c(t) + \lambda_l D_t, \qquad B_1^\ell(t) \triangleq -\lambda_l \frac{B_t^2}{R_t},$$
$$A_2^\ell(t) \triangleq -(Q_t H - \Pi_t D_t), \qquad B_2^\ell(t) \triangleq -A_c(t),$$

with terminal condition $Q^\ell = Q_T H$.

By Lemma 1, the unique solvability of (36) is guaranteed by the existence of a unique solution P^ℓ to the associated Riccati ODE on the interval $[0, T]$:

$$\begin{cases} -\dfrac{dP^\ell}{dt} = B_1^\ell(t)(P^\ell)^2 + (A_1^\ell(t) - B_2^\ell(t))P^\ell - A_2^\ell(t) \\ P^\ell(T) = Q_T H. \end{cases} \tag{45}$$

and therefore (36) admits a unique adapted solution $(\tilde{z}^{(\epsilon,\theta),\ell}, \tilde{s}^{(\epsilon,\theta),\ell}, \{q^{\ell,k}\})$, which is given by

$$\tilde{s}_t^{(\epsilon,\theta),\ell} = P^\ell(t)\tilde{z}_t^{(\epsilon,\theta),\ell}, \quad q_t^{\ell,k} = P^\ell(t)\kappa\epsilon\lambda_l\sigma^{\ell,k}. \tag{46}$$

with $\sigma^{\ell,k} \triangleq \sqrt{\nu_k}\langle\psi_k, \mathbf{f}_\ell\rangle$, and $\tilde{z}^{(\epsilon,\theta),\ell}$ is the solution to (42). □

Definition 6. The *perturbation response function* $e : [0,1] \times \mathcal{Q}_1 \to \mathbb{R}^+$ for the GMFG perturbation problem is defined as

$$e(\theta, \mathbf{Q}) = \sup_{\epsilon>0} \frac{\|\tilde{\mathbf{z}}^{(\epsilon,\theta)}\|_{L^2}}{\epsilon}. \tag{47}$$

Let $\tilde{A}^\ell(t) \triangleq A_c(t) + \lambda_l D_t - \lambda_l \frac{B_t^2}{R_t} P^\ell(t)$ and $\Psi^\ell(t,s)$ be the state transition function for this linear time-varying system, i.e.,

$$\Psi^\ell(t,s) \triangleq \exp\left(\int_s^t \tilde{A}^\ell(\tau)d\tau\right). \tag{48}$$

Theorem 3. *For the LQG-GMFG system, let the graphons $\mathbf{M} \in \mathcal{W}$ and $\mathbf{Q} \in \mathcal{Q}_1$ satisfy Assumption 1 with the eigenvalues $\{\lambda_\ell\}_{\ell=1}^d$ and $\{\nu_k\}_{k=1}^K$ arranged in descending order. Furthermore, assume the dynamic weighting terms $\{\gamma_\ell\}_{\ell=1}^d$ are in descending order where $\gamma_\ell \triangleq \lambda_\ell^2 \int_0^T \int_0^t [\Psi^\ell(t,s)]^2 ds dt$.*

The supremum of the perturbation response function $e(\theta, \cdot)$ is attained when $\psi_1 = \mathbf{f}_1$ and $\nu_1 = 1$, yielding

$$\sup_{\mathbf{Q}\in\mathcal{Q}_1} e(\theta, \mathbf{Q}) = \left\{\sum_{\ell=1}^d \lambda_\ell^2 \mathbf{f}_\ell^2(\theta) \int_0^T (\Psi^\ell(t,0))^2 dt + \kappa^2\gamma_1\right\}^{\frac{1}{2}} \tag{49}$$

Conversely, the infimum of its stochastic component is attained when $\psi_1 = \mathbf{f}_d$ and $\nu_1 = 1$.

Proof. By Lemma 1, we can explicitly obtain the solutions to $(\tilde{z}_t^{(\epsilon,\theta),\ell}, \tilde{s}_t^{(\epsilon,\theta),\ell})$ where $\tilde{z}_t^{(\epsilon,\theta),\ell}$ is given by

$$\tilde{z}_t^{(\epsilon,\theta),\ell} = \Psi^\ell(t,0)\tilde{z}_0^{(\epsilon,\theta),\ell} + \kappa\epsilon\lambda_\ell \sum_{k=1}^K \int_0^t \Psi^\ell(t,s)\sigma^{\ell,k} dW_k(s). \tag{50}$$

We split $\tilde{z}_t^{(\epsilon,\theta),\ell} = \tilde{z}_{t,\text{det}}^{(\epsilon,\theta),\ell} + \tilde{z}_{t,\text{stoch}}^{(\epsilon,\theta),\ell}$ as follows:

- Deterministic part $\tilde{z}_{t,det}^{(\epsilon,\theta),\ell} = \Psi^\ell(t,0)\lambda_l f_l(\theta)\epsilon$;
- Stochastic part

$$\tilde{z}_{t,stoch}^{(\epsilon,\theta),\ell} = \kappa\epsilon\lambda_\ell \sum_{k=1}^K \int_0^t \Psi^\ell(t,s)\sigma^{\ell,k} dW_k(s).$$

This leads to $\mathbb{E}[(\tilde{z}_t^{(\epsilon,\theta),\ell})^2] = (\tilde{z}_{t,\text{det}}^{(\epsilon,\theta),\ell})^2 + \mathbb{E}[(\tilde{z}_{t,\text{stoch}}^{(\epsilon,\theta),\ell})^2]$ where

$$\begin{aligned} \mathbb{E}[(\tilde{z}_{t,\text{stoch}}^{(\epsilon,\theta),\ell})^2] &= \text{var}(\tilde{z}_{t,\text{stoch}}^{(\epsilon,\theta),\ell}) \\ &= \kappa^2\epsilon^2\lambda_l^2 \sum_{k=1}^K (\sigma^{\ell,k})^2 \int_0^t (\Psi^\ell(t,s))^2 ds \\ &= \kappa^2\epsilon^2\lambda_l^2 \sum_{k=1}^K \nu_k |\langle\psi_k, f_l\rangle|^2 \int_0^t (\Psi^\ell(t,s))^2 ds, \end{aligned}$$

then the L^2-norm of $\tilde{\mathbf{z}}^{(\epsilon,\theta)}$ satisfies

$$\begin{aligned} \|\tilde{\mathbf{z}}^{(\epsilon,\theta)}\|_{L^2} &= \left\{\int_0^T \int_0^1 \mathbb{E}\left[(\tilde{\mathbf{z}}^{(\epsilon,\theta)})^2(t,\beta)\right] d\beta dt\right\}^{\frac{1}{2}} \\ &= \left\{\int_0^T \int_0^1 \sum_{\ell=1}^d \mathbf{f}_\ell^2(\beta)\mathbb{E}[(\tilde{z}_t^{(\epsilon,\theta),\ell})^2] d\beta\, dt\right\}^{\frac{1}{2}} \\ &= \left\{\int_0^T \int_0^1 \sum_{\ell=1}^d \mathbf{f}_\ell^2(\beta)\left((\tilde{z}_{t,\text{det}}^{(\epsilon,\theta),\ell})^2 + \mathbb{E}[(\tilde{z}_{t,\text{stoch}}^{(\epsilon,\theta),\ell})^2]\right) d\beta\, dt\right\}^{\frac{1}{2}} \end{aligned}$$

where

$$\int_0^T \int_0^1 \sum_{\ell=1}^d \mathbf{f}_\ell^2(\beta)\left(\tilde{z}_{t,\text{det}}^{(\epsilon,\theta),\ell}\right)^2 d\beta\, dt = \epsilon^2 \sum_{\ell=1}^d \lambda_\ell^2 \mathbf{f}_\ell^2(\theta) \int_0^T (\Psi^\ell(t,0))^2 dt$$

and

$$\begin{aligned} &\int_0^T \int_0^1 \sum_{\ell=1}^d \mathbf{f}_\ell^2(\beta)\mathbb{E}[(\tilde{z}_{t,\text{stoch}}^{(\epsilon,\theta),\ell})^2] d\beta\, dt \\ &= \kappa^2\epsilon^2 \sum_{\ell=1}^d \lambda_\ell^2 \int_0^T \int_0^t [\Psi^\ell(t,s)]^2 ds dt \sum_{k=1}^K \nu_k |\langle\psi_k, \mathbf{f}_\ell\rangle|^2 \\ &= \kappa^2\epsilon^2 \sum_{\ell=1}^d \gamma_\ell \sum_{k=1}^K \nu_k |\langle\psi_k, \mathbf{f}_\ell\rangle|^2, \end{aligned} \tag{51}$$

then (51) reaches its supremum subject to the constraint that $\|\mathbf{Q}\|_2 = 1$ if and only if $\psi_1 = \mathbf{f}_1, \nu_1 = 1$ and (51) is reduced to $\kappa^2\epsilon^2\gamma_1$.

The results for the infimum of (51) for some $k > 1$ can be obtained analogously. Hence, the worst-case $e(\theta, \cdot)$ is explicitly given by (49) as required. □

IV. Simulation Results

Consider the following parameters: $A = 0.5$, $B = 1$, $D = 1$, $H = 1$, $Q_t = Q_T = 2$, $R = 1$, $T = 1$, $\kappa = 1$, and the discretized time-step $dt = 0.01$, with a spectral decomposition $\lambda_\ell = \frac{4}{\ell^2\pi^2}$ and $\mathbf{f}_\ell(x) = \sqrt{2}\cos\left(\frac{\pi\ell x}{2}\right)$ of $\mathbf{M}$ (see Fig. 1) where ℓ is an odd integer [8]. In our experiment, $\ell \leq d = 11$.

By Theorem 3, the function $e(\theta, \cdot)$ reaches its maximum when the dominant eigenfunction of the perturbation graphon $\mathbf{Q}$ aligns with the dominant eigenfunction of $\mathbf{M}$, i.e., $\mathbf{f}_1$. Conversely, $e(\theta, \cdot)$ is minimized when the $\mathbf{M}$-eigenfunctions corresponding to large eigenvalues align with the $\mathbf{Q}$-eigenfunctions corresponding to small eigenvalues. To

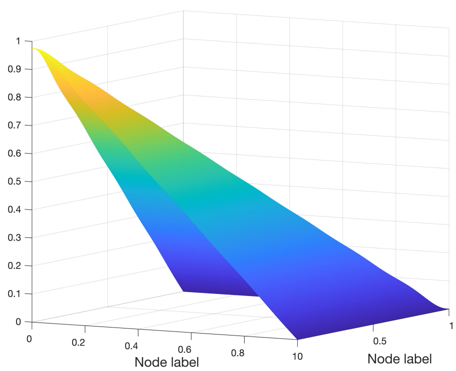

Fig. 1. Graphon $\mathbf{M}(x, y) = 1 - \max(x, y), x, y \in [0, 1]$.

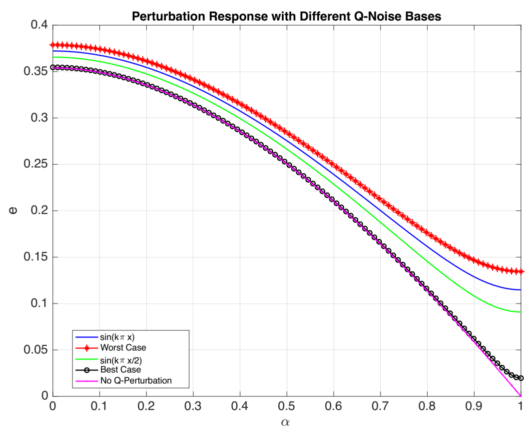

Fig. 2. Perturbation response with the Uniform Attachment graphon and different Q-noise bases.

illustrate our findings on the worst-case and best-case $\mathbf{Q}$-perturbations and initial state perturbations, we compare the response functions for different $\mathbf{Q} \in \mathcal{Q}_1$. The comparison is shown in Fig. 2, which illustrates that the alignment of eigenfunctions between the network graphon $\mathbf{M}$ and the perturbation graphon $\mathbf{Q}$ determines the upper and lower bounds of the perturbation response. Additionally, our analysis identifies nodes 0 and 1 as critical nodes, with node 0 being the most significant and node 1 the least significant. This aligns with node 0's central position in the network, as indicated by interpreting $\mathbf{M}(\alpha, \beta)$ as the connection density between nodes.

V. Conclusion

This work studies LQG-GMFGs by integrating initial state perturbation and Q-noise perturbations for large-scale network systems. Employing spectral decomposition methods, explicit expressions are obtained for the perturbation response, revealing how network topology and noise covariance interplay with system dynamics. The results show that the perturbation response depends on the alignment of eigenfunctions between the network and noise structures, specifically, the initial state perturbation component of (49) is closely related to the eigenfunction $\mathbf{f}$ of the network graphon $\mathbf{M}$, while the Q-noise perturbation part depends on the alignment of the maximum/minimum eigenfunctions of $\mathbf{M}$ and $\mathbf{Q}$. This is analogous to findings in the context of long-range average optimal control for Q-noise graphon systems, where the maximum performance loss is also achieved when the Q-noise covariance operator aligns with the dominant eigenmode of the system's dynamics operator [11]. Future work shall explore dynamic topologies and nonlinear dynamics to extend its applicability to complex scenarios.

VI. Acknowledgment

The authors would like to thank Alex Dunyak for valuable suggestions concerning this work.

References

[1] M. Huang, R. P. Malhamé, and P. E. Caines, "Large population stochastic dynamic games: Closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle," *Commun. Inf. Syst.*, vol. 6, no. 3, pp. 221–252, 2006.

[2] J.-M. Lasry and P.-L. Lions, "Mean field games," *Jpn. J. Math.*, vol. 2, no. 1, pp. 229–260, 2007.

[3] D. Firoozi, A. Pakniyat, and P. E. Caines, "A mean field game – hybrid systems approach to optimal execution problems in finance with stopping times," in *Proc. 56th IEEE Conf. Decision and Control (CDC)*. IEEE, 2017, pp. 3144–3151.

[4] K. Huang, X. Chen, X. Di, and Q. Du, "Dynamic driving and routing games for autonomous vehicles on networks: A mean field game approach," *Transp. Res. C, Emerg. Technol.*, vol. 128, p. 103189, 2021.

[5] R. Carmona, M. Laurière, and Z. Tan, "Model-free mean-field reinforcement learning: Mean-field MDP and mean-field Q-learning," *Ann. Appl. Probab.*, vol. 33, no. 6B, pp. 5334–5381, 2023.

[6] M. Laurière, S. Perrin, M. Geist, and O. Pietquin, "Learning mean field games: A survey," 2022, arXiv preprint arXiv:2205.12944.

[7] P. E. Caines and M. Huang, "Graphon mean field games and the GMFG equations: ϵ-Nash equilibria," in *Proc. 58th IEEE Conf. Decision and Control (CDC)*. IEEE, 2019, pp. 286–292.

[8] S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: Analysis via graphon-invariant subspaces," *IEEE Trans. Autom. Control*, vol. 68, no. 12, pp. 7482–7497, 2023.

[9] P. E. Caines and M. Huang, "Graphon mean field games and their equations," *SIAM J. Control Optim.*, vol. 59, no. 6, pp. 4373–4399, 2021.

[10] A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with Q-space noise," in *Proc. 61st IEEE Conf. Decision and Control (CDC)*. IEEE, 2022, pp. 3926–3932.

[11] ——, "Quadratic optimal control of graphon Q-noise linear systems," 2024, arXiv preprint arXiv:2407.00212. To appear in *Automatica*.

[12] T. Zhang, S. Gao, and P. E. Caines, "Sensitivity analysis for network LQG mean-field games: A graphon limit approach," in *10th IFAC Conf. Networked Systems (NECSYS)*, vol. 59, no. 4, 2025, pp. 121–126.

[13] D. Xu, Z. Gou, N. Huang, and S. Gao, "Linear-quadratic graphon mean field games with common noise," 2024, arXiv preprint arXiv:2401.09030.

[14] L. Lovász, *Large networks and graph limits*. Amer. Math. Soc., 2012, vol. 60.

[15] J. Ma and J. Yong, *Forward-backward stochastic differential equations and their applications*, ser. Stochastic Modelling and Applied Probability. Springer Science & Business Media, 1999, vol. 1702.

[16] A. X. Jiang and K. Leyton-Brown, "A tutorial on the proof of the existence of Nash equilibria," University of British Columbia, Tech. Rep. TR-2007-25, 2009.

[17] S. Peng and Z. Wu, "Fully coupled forward-backward stochastic differential equations and applications to optimal control," *SIAM J. Control Optim.*, vol. 37, no. 3, pp. 825–843, 1999.