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LQG Mean Field Games with Covariance-Matrix-Dependent Cost Coefficients ⋆

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Abstract: This paper studies linear quadratic Gaussian (LQG) Mean Field Games (MFGs) where coefficients of quadratic cost functions depend on the covariance matrix of the population's state distribution. Such formulations allow for modelling agents whose costs are not only sensitive to the instantaneous population state average but also the population state dispersion, which serves as a measure of instantaneous risk. The calculation of the possible MFG equilibria involves solving (i) two nonlinearly coupled differential equations (one Riccati equation and the other a differential Lyapunov equation for the covariance matrix evolution) and (ii) an additional Riccati equation for the local feedback gain. Sufficient conditions for the existence and the uniqueness of an MFG equilibrium solution are established.

Keywords: Mean field games, Riccati equation, Lyapunov equation

1. INTRODUCTION

The mean field game (MFG) theory (Huang et al., 2006; Lasry and Lions, 2006) provides a theoretical framework to identify decentralized approximate solutions to large-population stochastic games. MFG finds applications in domains such as finance (Casgrain and Jaimungal, 2020), energy systems (Kizilkale et al., 2019), transportation dynamics (Huang et al., 2021), and epidemic modelling (Aurell et al., 2022). Variations and extensions of MFG have been investigated using techniques from adaptive control, reinforcement learning, risk-sensitive control, or robust control (Subramanian and Mahajan, 2019; Laurière et al., 2022; Tembine et al., 2013; Bauso et al., 2016; Kizilkale and Caines, 2012).

In MFG problems, higher order moments can be considered in the problem formulation. One variation of MFG problems considering higher-order moments incorporates the quantile of the mean field distribution (see (Tembine, 2017; Tchuendom et al., 2019; Foguen Tchuendom et al., 2024; Gao and Malhamé, 2025; Tchuendom et al., 2025)). The quantile function allows a simple representation of the mean-field distribution involving higher order moments (e.g., variances in the Gaussian case) and the flexibility of adjusting the distribution couplings via the quantile parameter (Tchuendom et al., 2019; Foguen Tchuendom et al., 2024; Tchuendom et al., 2025). The linear quadratic Gaussian (LQG) MFGs where the cost coefficients depend on the quantile of the state distribution was investigated in (Gao and Malhamé, 2025) where each agent has a scalar state. Due to the Gaussian nature of the LQG MFG problem, the quantile can be represented by the mean and variance of the state distribution, resulting in LQG MFG problems where cost coefficients depend on the mean and variance of the state distribution.

The cost coefficients of MFG problems can be designed to depend on population behavior. One closely related work with such non-standard cost coefficients is the integral control based MFG (Kizilkale et al., 2019) where the coefficient in the running cost depends on an integral term related to the population behavior over time. The dependence of the cost coefficient on the population in (Kizilkale et al., 2019) differs from the current paper. The work (Fischer and Livieri, 2016) treats the mean-variance portfolio optimization problems using a mean field approach, and hence is different from the current paper in the problem formulation.

The main contribution of this work includes (a) generalizing LQG MFGs with variance-dependent coefficients to the multi-dimensional state setting (compared to the work (Gao and Malhamé, 2025)), (b) establishing sufficient conditions for the existence and the uniqueness of solutions to a new pair of nonlinearly coupled matrix differential equations (consisting of one Riccati equation and the other a differential Lyapunov equation for the covariance matrix evolution), required to characterize the MFG solutions, and (c) establishing sufficient conditions for the existence and uniqueness of the resulting MFG solutions.

Notation: We use $\|\cdot\|$ without subscript to denote both the vector 2-norm and the induced matrix 2-norm. Let $\mathbb{R}$ denote real numbers. Let $C([0,T],\mathbb{R}^n)$ denote the set of continuous functions from $[0,T]$ to $\mathbb{R}^n$. For $x\in C([0,T],\mathbb{R}^n)$, $\|x\|_\infty := \sup_{t\in[0,T]}\|x(t)\|$ denotes the uniform norm. For $x\in\mathbb{R}^n$ and $Q\in\mathbb{R}^{n\times n}$, let $\|x\|_Q^2 := x^\intercal Q x$. Let I_n denote the n-dimensional identity matrix.

2. PROBLEM STATEMENT

2.1 Finite Agent Stochastic Games

Consider the following linear quadratic stochastic games with N agents: the dynamics of agent $i\in\{1,\cdots,N\}$ are given by

$$dx_i(t) = (Ax_i(t) + Bu_i(t))dt + \Sigma dw_i(t), \qquad (1)$$

with $x_i(t)\in\mathbb{R}^n, u_i(t)\in\mathbb{R}^m, w_i(t)\in\mathbb{R}^q, t\in[0,T]$, and the cost that agent i aims to minimize is given by

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$$J_i(u_i, u_{-i}) = \mathbb{E}\int_0^T \left(\|x_i(t) - \bar{x}(t)\|^2_{Q_t^{eq}} + \|u_i(t)\|^2_R\right) dt + \mathbb{E}\|x_i(T) - \bar{x}(T)\|^2_{Q_T}, \quad R > 0, \tag{2}$$

where $x_i(t)$ (resp. $u_i(t)$) is the state (resp. the control action) of agent i at time $t \in [0, T]$, u_i denotes the strategy of agent i, u_{-i} denotes the strategies of all other agents except agent i, $w_i(t)$ is the standard q-dimensional Brownian motion, the initial $x_i(0) \sim \mathcal{N}(\mu_0, V_0)$ is assumed to be Gaussian with covariance $V_0 > 0$, and the average of the population state is denoted by

$$\bar{x}^N(t) := \frac{1}{N}\sum_{i=1}^N x_i(t).$$

We assume that the initial conditions $\{x_i(0) : 1 \leq i \leq n\}$ are independent, the Brownian motions $\{w_i(\cdot) : 1 \leq i \leq n\}$ are independent among agents, and also independent of the initial conditions. The cost coefficient matrix depends on the covariance matrix as follows:

$$Q_t^{eq} = \Theta(V^N(t)) \geq 0, \quad \forall t \in [0, T), \tag{3}$$

where the function $\Theta : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ determines the cost structures and weights based on the covariance matrix

$$V^N(t) := \mathbb{E}(x_i(t) - \bar{x}^N(t))(x_i(t) - \bar{x}^N(t))^\intercal.$$

Depending on the function Θ, the cost function could encourage or discourage large instantaneous dispersions in the population state distribution. The decentralized admissible control u_i for agent i is adapted to the filtration $\{\mathcal{F}_t^i\}_{0\leq t\leq T}$ with $\mathcal{F}_t^i := \sigma(x_i(0), w_i(s) : 0 \leq s \leq t)$, generated by its initial condition and its Brownian motion, and $\mathbb{E}\int_0^T u_i^\intercal(t)u_i(t)dt < \infty$.

2.2 Limit MFG Problems

To identify approximate decentralized solutions for the problem above when the population size is large, we apply the MFG approach (Huang et al., 2007) by taking the population limit and obtain approximate solutions via solving the limiting MFG problem. The corresponding LQG mean field tracking problem in the large population limit for a generic agent i is given by the dynamics

$$dx_i(t) = (Ax_i(t) + Bu_i(t))dt + \Sigma dw_i(t), \tag{4}$$

with the initial condition $x_i(0) \sim \mathcal{N}(\mu_0, V_0)$ and the covariance-dependent cost that the generic agent aims to minimize

$$J_i(u_i, \bar{x}) = \mathbb{E}\int_0^T \left(\|x_i(t) - \bar{x}(t)\|^2_{Q_t^{eq}} + \|u_i(t)\|^2_R\right) dt + \mathbb{E}\|x_i(T) - \bar{x}(T)\|^2_{Q_T}, \quad R > 0, \tag{5}$$

where

$$\bar{x}(t) \triangleq \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N x_i(t)$$

and the covariance-dependent coefficient

$$Q_t^{eq} = \Theta(V(t)) \geq 0, \quad \forall t \in [0, T). \tag{6}$$

The function $\Theta(\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ determines the cost structures and weights based on the covariance matrix of the states

$$V(t) := \mathbb{E}(x_i(t) - \bar{x}(t))(x_i(t) - \bar{x}(t))^\intercal.$$

It is assumed that the initial mean μ_0 and initial covariance matrix V_0 are known, and $\|V_0\| > 0$. Throughout the paper, the time index is omitted when there is no risk of confusion.

Remark 1. (Examples of Θ). (a) In the case where $\Theta(V(t)) \triangleq V(t)^{-1} > 0$, the quadratic running cost in states becomes

$$\mathbb{E}\|x_i - \bar{x}\|^2_{Q_t^{eq}} = \text{Tr}(V(t)^{-1}\mathbb{E}(x_i - \bar{x})(x_i - \bar{x})^\intercal) = n,$$

and hence the problem is equivalent to a problem with only the control cost and the terminal cost. (b) In the case where $n = 1$ and $\Theta(V(t)) \triangleq \sqrt{V(t)}\Theta^{-1}_{(\alpha)} > 0$ where $\Theta^{-1}_{(\cdot)}$ is the probit function, i.e. the inverse cumulative distribution of the standard scalar-valued Gaussian distribution, the cost depends on the quantile of $x_i - \bar{x}$ (as in (Gao and Malhamé, 2025)).

3. MFG SOLUTIONS

We follow the fixed-point approach (e.g. (Huang et al., 2007)) to derive the MFG solutions. First, given the mean field (Gaussian) distribution which is completely characterized by $(\bar{x}(t), V(t))_{t\in[0,T]}$, by dynamic programming, the optimal tracking control (i.e. the best response) for agent i satisfies

$$u_i^*(t) = -R^{-1}B^\intercal(\Pi_t x_i(t) + s_t) \tag{7}$$

where

$$-\dot{\Pi}_t = A^\intercal\Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\intercal\Pi_t + Q_t^{eq}, \quad \Pi_T = Q_T,$$
$$-\dot{s}_t = \left(A - BR^{-1}B^\intercal\Pi_t\right)^\intercal s_t - Q_t^{eq}\bar{x}(t), \quad s_T = -Q_T\bar{x}(T).$$

Under the best response (7), the closed-loop dynamics for a generic agent i are then given by

$$dx_i = (A - BR^{-1}B^\intercal\Pi_t)x_i dt - BR^{-1}B^\intercal s_t dt + \Sigma dw_i.$$

Taking the expectation over the corresponding integral equation yields the evolution of the mean, given by

$$d\bar{x}_i = \left((A - BR^{-1}B^\intercal\Pi_t)\bar{x}_i - BR^{-1}B^\intercal s_t\right) dt, \ \bar{x}_i(0) = \mu_0.$$

We note that $\bar{x}_i = \bar{x}$ for all i, since agents are assumed here to have homogeneous parameters. Let $e_i = x_i - \bar{x}_i$. Then

$$de_i = (A - BR^{-1}B^\intercal\Pi_t)e_i dt + \Sigma dw_i.$$

Applying Itô's lemma for $e_i e_i^\intercal$ and taking the expectation yields the evolution of the covariance $V(t) = \mathbb{E}[e_i e_i^\intercal]$, governed by the following Lyapunov equation

$$\dot{V}(t) = (A - BR^{-1}B^\intercal\Pi_t)V(t) + V(t)(A - BR^{-1}B^\intercal\Pi_t)^\intercal + \Sigma\Sigma^\intercal$$

with the initial condition $V(0) = V_0$. See e.g. (Åström, 2012) for the covariance evolution for linear stochastic differential equations.

Therefore, we have the following lemma.

Lemma 1. (MFG Solutions). The solution to the limit MFG problem specified by (4), (5) and (6), if exists, satisfies

$$u_i^*(t) = -R^{-1}B^\intercal(\Pi_t x_i(t) + s_t), \quad t \in [0, T] \tag{8}$$

where

$$-\dot{\Pi}_t = A^\intercal\Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\intercal\Pi_t + Q_t^{eq}, \quad \Pi_T = Q_T$$
$$-\dot{s}_t = \left(A - BR^{-1}B^\intercal\Pi_t\right)^\intercal s_t - Q_t^{eq}\bar{x}(t), \quad s_T = -Q_T\bar{x}(T)$$
$$\dot{\bar{x}}(t) = (A - BR^{-1}B^\intercal\Pi_t)\bar{x}(t) - BR^{-1}B^\intercal s_t, \quad \bar{x}(0) = \mu_0$$
$$\dot{V}(t) = (A - BR^{-1}B^\intercal\Pi_t)V(t) + V(t)(A - BR^{-1}B^\intercal\Pi_t)^\intercal + \Sigma\Sigma^\intercal, \quad V(0) = V_0$$

with the constraint

$$Q_t^{eq} = \Theta(V(t)), \quad \forall t \in [0, T). \tag{9}$$

□

Since the mean of the state distribution is not involved in Q_t^{eq}, the dynamics of s and $\bar{x}$ can be decoupled via a Riccati equation as follows. We first introduce the ansatz $s_t = P_t\bar{x}(t)$ to decouple the differential equations for s and $\bar{x}$. Then

$$\begin{aligned}-\dot{P}_t\bar{x}(t) &= P_t\dot{\bar{x}}(t) - \dot{s}_t\\ &= P_t[(A - BR^{-1}B^\intercal\Pi_t)\bar{x} - BR^{-1}B^\intercal P_t\bar{x}]\\ &\quad + \big(A - BR^{-1}B^\intercal\Pi_t\big)^\intercal P_t\bar{x} - Q_t^{eq}\bar{x}\\ &= \big[(A - BR^{-1}B^\intercal\Pi_t)^\intercal P_t + P_t(A - BR^{-1}B^\intercal\Pi_t)\\ &\quad - P_tBR^{-1}B^\intercal P_t - Q_t^{eq}\big]\bar{x}.\end{aligned}$$

Since this should hold for all $\bar{x}$, we obtain the following Riccati equation that decouples the dynamics for $\bar{x}$ and s:

$$\begin{aligned}-\dot{P}_t &= (A - BR^{-1}B^\intercal\Pi_t)^\intercal P_t + P_t(A - BR^{-1}B^\intercal\Pi_t)\\ &\quad - P_tBR^{-1}B^\intercal P_t - Q_t^{eq}, \quad P_T = -Q_T.\end{aligned}$$

Thus, the MFG solutions are simplified to the following.

Lemma 2. (MFG Solutions via Riccati Equation). The solution to the limit MFG problem specified by (4), (5) and (6), if exists, satisfies

$$u_i^*(t) = -R^{-1}B^\intercal(\Pi_t x_i(t) + P_t\bar{x}(t)) \tag{10}$$

where $\bar{x}$ satisfies

$$\dot{\bar{x}}(t) = (A - BR^{-1}B^\intercal(\Pi_t + P_t))\bar{x}(t), \quad \bar{x}(0) = \mu_0,$$

and $\Pi_{(\cdot)}$, $P_{(\cdot)}$ and $V(\cdot)$ satisfy

$$-\dot{\Pi}_t = A^\intercal\Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\intercal\Pi_t + Q_t^{eq},\ \Pi_T = Q_T \tag{11}$$

$$\begin{aligned}\dot{V}(t) &= (A - BR^{-1}B^\intercal\Pi_t)V(t)\\ &\quad + V(t)(A - BR^{-1}B^\intercal\Pi_t)^\intercal + \Sigma\Sigma^\intercal, V(0) = V_0\end{aligned} \tag{12}$$

$$\begin{aligned}-\dot{P}_t &= (A - BR^{-1}B^\intercal\Pi_t)^\intercal P_t + P_t(A - BR^{-1}B^\intercal\Pi_t)\\ &\quad - P_tBR^{-1}B^\intercal P_t - Q_t^{eq}, \quad P_T = -Q_T,\end{aligned} \tag{13}$$

with the constraint $Q_t^{eq} = \Theta(V(t)) \geq 0$. □

Remark 2. The MFG solution involves solving three coupled matrix differential equations: two Riccati equations specified by (13) and (11), and one Lyapunov equation (12). We note that the two equations (11) and (12) are coupled, and the equation (13) can be solved after the first two equations (11) and (12) are solved. □

The solution can be further simplified by introducing $H_t := \Pi_t + P_t$ for all $t \in [0, T]$. Summing both sides of the differential equations for Π and P yields

$$-\dot{H}_t = A^\intercal H_t + H_t A - H_t BR^{-1}B^\intercal H_t, \quad H_T = 0. \tag{14}$$

Clearly $H_t = 0$ for all $t \in [0, T]$.

Based on this simplification above, the MFG best response solution is equivalently given by

$$\begin{aligned}u_i^*(t) &= -R^{-1}B^\intercal(\Pi_t x_i(t) + (H_t - \Pi_t)\bar{x}(t))\\ &= -R^{-1}B^\intercal\Pi_t(x_i(t) - \bar{x}(t)).\end{aligned} \tag{15}$$

Lemma 3. (Simplified MFG Solutions). The solution to the limit MFG problem specified by (4), (5) and (6), if exists, satisfies

$$u_i^*(t) = -R^{-1}B^\intercal\Pi_t(x_i(t) - \bar{x}(t)) \tag{16}$$

where

$$\begin{aligned}\dot{V}(t) &= (A - BR^{-1}B^\intercal\Pi_t)V(t)\\ &\quad + V(t)(A - BR^{-1}B^\intercal\Pi_t)^\intercal + \Sigma\Sigma^\intercal, V(0) = V_0\end{aligned} \tag{17}$$

$$-\dot{\Pi}_t = A^\intercal\Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\intercal\Pi_t + \Theta(V(t)), \Pi_T = Q_T \tag{18}$$

and $\bar{x}$ satisfies

$$\dot{\bar{x}}(t) = A\bar{x}(t), \quad \bar{x}(0) = \mu_0.$$

□

Remark 3. In some cases, the MFG solutions do not permit the simplification in Lemma 3. Such cases include:

(i) the case where there is a mean field term in the dynamics, the decomposition does not hold, since there will be an additional term of Π_t in the dynamics of H_t;
(ii) the case where there is a coefficient in front of the mean field term in the cost that is not identity (that is in the cost there are terms $\|x_i - \Gamma\bar{x}\|^2_{Q_t^{eq}}$ with $\Gamma \neq I$), the cancellation of Q^{eq} will not hold when summing both sides of the differential equations for Π and P.

In fact, the two cases above correspond to the cases for which the mean field type control solution (Bensoussan et al., 2013) does not coincide with the limit MFG solution (see (Bensoussan et al., 2013, Section 6)). Moreover, it is worth highlighting that for the two cases above, the mean field type control problem can be solved via a decomposition method similar to the network-coupled control (Gao and Mahajan, 2021) with rank-one network coupling, and the resulting control gain (16) resembles that in (Gao and Mahajan, 2021, Remark 1). □

4. SUFFICIENT CONDITIONS FOR EXISTENCE AND UNIQUENESS OF SOLUTIONS

4.1 *Sufficient Condition for Existence*

In the following, we establish sufficient conditions under which the new pair of coupled equations (17) and (18) have a solution pair (Π, V). We introduce the following assumptions.

Assumption 1. (A1). $\Theta(E) \geq 0$, for all $E \in \mathbb{R}^{n\times n}$, and $\Theta(\cdot)$ is continuous with respect to its argument.

Assumption 2. (A2). $\Theta(\cdot)$ satisfies that for all $E \in \mathbb{R}^{n\times n}$ with $0 < b_1 \leq \|E\| \leq b_2 < \infty$, the following holds

$$\|\Theta(E)\| \leq c_1 + c_2(b_1, b_2)\|E\|,$$

for some $c_1 \in \mathbb{R}$, and $0 \leq c_2(b_1, b_2) < \infty$.

Remark 4. Examples of $\Theta(E)$ that satisfies the assumption (A2) include $\exp(E)$, E^{-1} and $\sum_{k=0}^{K} E^k$. □

Consider the Banach space of continuous functions $C([0, T], \mathbb{R}^n)$ with the uniform norm $\|\cdot\|_\infty$. Given $\Pi_{(\cdot)} \in C([0, T], \mathbb{R}^n)$, the solution to (17) satisfies

$$V(t) = \Phi(t, 0)V_0\Phi^\intercal(t, 0) + \int_0^t \Phi(t, s)\Sigma\Sigma^\intercal\Phi^\intercal(t, s)\, ds \tag{19}$$

where $\Phi(t, s)$ is the state transition matrix that satisfies

$$\frac{\partial}{\partial t}\Phi(t, s) = (A - BR^{-1}B^\intercal\Pi_t)\Phi(t, s), \quad \Phi(s, s) = I, \tag{20}$$

for all $t, s \in [0, T]$.

Denote $\|A\| = \alpha$, $\|B\| = \beta$, $\|R\| = r > 0$.

Proposition 1. (Existence). Consider $[0, T]$ and $K := \{\Pi \in C([0, T], \mathbb{R}^n) : \|\Pi\|_\infty \leq M\}$ with $M > 0$. Assume (A1)-(A2) hold. If T and M satisfy the following inequality

$$\begin{aligned}&T\big[(2\alpha M + \frac{\beta^2}{r}M^2 + c_1)\\ &\quad + c_2(b_1, b_2)e^{2(\alpha + \frac{\beta^2}{r}M)T}\big(\|V_0\| + T\|\Sigma\Sigma^\intercal\|\big)\big] \leq M,\end{aligned} \tag{21}$$

with the coefficients in the assumption (A2) given by $b_1 = e^{-2(\alpha + \frac{\beta^2}{r}M)T}\|V_0\|$ and $b_2 = e^{2(\alpha + \frac{\beta^2}{r}M)T}\big(\|V_0\| + T\|\Sigma\Sigma^\intercal\|\big)$, then the coupled differential equations (17) and (18) with the coupling constraint (6) have at least one solution pair (V, Π) with $\Pi \in K$ and $V \in C([0, T], \mathbb{R}^n)$. □

PROOF Substituting $V(t)$ in (18) using the representation in (19) yields

$$-\dot{\Pi}_t = A^\intercal \Pi_t + \Pi_t A - \Pi_t B R^{-1} B^\intercal \Pi_t + \Theta\Big(\Phi(t,0) V_0 \Phi^\intercal(t,0) + \int_0^t \Phi(t,s)\Sigma\Sigma^\intercal \Phi^\intercal(t,s)\, ds\Big).$$

The equivalent integral representation is

$$\Pi_t = \Pi_T + \int_t^T \Big[A^\intercal \Pi_q + \Pi_q A - \Pi_q B R^{-1} B^\intercal \Pi_q + \Theta\Big(\Phi(q,0) V_0 \Phi^\intercal(q,0) + \int_0^q \Phi(q,s)\Sigma\Sigma^\intercal \Phi^\intercal(q,s)\, ds\Big)\Big] dq.$$

Let $\mathcal{T}$ denote the operator on the Banach space of continuous functions $C([0,T],\mathbb{R}^n)$ with the uniform norm $\|\cdot\|_\infty$, defined as follows: for any $\Pi \in C([0,T],\mathbb{R}^n)$,

$$\mathcal{T}(\Pi)(t) = \int_t^T \Big[A^\intercal \Pi_q + \Pi_q A - \Pi_q B R^{-1} B^\intercal \Pi_q + \Theta\Big(\Phi(q,0) V_0 \Phi^\intercal(q,0) + \int_0^q \Phi(q,s)\Sigma\Sigma^\intercal \Phi^\intercal(q,s)\, ds\Big)\Big] dq \tag{22}$$

for all $t \in [0,T]$. Given $M > 0$, consider the set of uniformly bounded continuous function over $[0,T]$ defined by

$$K := \{\Pi \in C([0,T],\mathbb{R}^n) : \|\Pi\|_\infty \le M\}.$$

Then it is easy to verify that K is a nonempty, bounded, convex and closed subset of the Banach space $C([0,T],\mathbb{R}^n)$ with the uniform norm $\|\cdot\|_\infty$.

For any element $\Pi \in K$, applying Lemma 4 and Lemma 5 (see Appendix), we obtain the following: for any $t \in [0,T]$

$$\begin{aligned}
\|\mathcal{T}(\Pi)(t)\|_\infty \le{}& T(2\alpha M + \frac{\beta^2}{r} M^2) \\
&+ T \sup_{q\in[0,T]} \Big\|\Theta\Big(\Phi(q,0)V_0\Phi^\intercal(q,0) + \int_0^q \Phi(q,s)\Sigma\Sigma^\intercal\Phi^\intercal(q,s)\, ds\Big)\Big\| \\
&\text{(by Lemma 5 and (A2))} \\
\le{}& T(2\alpha M + \frac{\beta^2}{r} M^2) + Tc_1 + Tc_2(b_1,b_2) \sup_{q\in[0,T]} \Big\|\Phi(q,0)V_0\Phi^\intercal(q,0) + \int_0^q \Phi(q,s)\Sigma\Sigma^\intercal\Phi^\intercal(q,s)\, ds\Big\| \\
\le{}& T(2\alpha M + \frac{\beta^2}{r} M^2 + c_1) + Tc_2(b_1,b_2) \sup_{q\in[0,T]} \Big(e^{2(\alpha+\frac{\beta^2}{r}M)q}\|V_0\| + \int_0^q e^{2(\alpha+\frac{\beta^2}{r}M)(q-s)}\|\Sigma\Sigma^\intercal\|\, ds\Big) \\
\le{}& T(2\alpha M + \frac{\beta^2}{r} M^2 + c_1) + Tc_2(b_1,b_2) e^{2(\alpha+\frac{\beta^2}{r}M)T}\Big(\|V_0\| + T\|\Sigma\Sigma^\intercal\|\Big).
\end{aligned}$$

Then if (21) holds, we obtain

$$\|\mathcal{T}(\Pi)\|_\infty := \sup_{t\in[0,T]} \|\mathcal{T}(\Pi)(t)\| \le M.$$

We have established that $\{\mathcal{T}(\Pi) : \Pi \in K\}$ under the condition (21) above is uniformly bounded. Furthermore, one can show that $\{\mathcal{T}(\Pi) : \Pi \in K\}$ is equicontinuous (see Lemma 6). Then by Arzelà-Ascoli Theorem (Conway, 1990, p.175), the set $\{\mathcal{T}(\Pi) : \Pi \in K\}$ is totally bounded and hence the closure of $\{\mathcal{T}(\Pi) : \Pi \in K\}$ is compact (Rudin et al., 1976, Appendix A4). By definition (Conway, 1990, Def. 4.1), $\mathcal{T} : K \to K$ is a compact operator. Since K is a closed, bounded, and convex subset of the normed space $C([0,T];\mathbb{R}^n)$ endowed with the uniform norm and $\mathcal{T} : K \to C([0,T];\mathbb{R}^n)$ is a compact map such that $\{\mathcal{T}(\Pi) : \Pi \in K\} \subseteq K$ (under the condition (21)), an application of the Schauder fixed-point theorem (Conway, 1990, p. 150) implies that the equation (11) has at least one solution $\Pi \in K$.

Then, given $\Pi \in K$, the equation (17) is a linear time-varying differential equation with continuous coefficients and hence the solution exists. Therefore the coupled differential equations (18) and (17) have at least one joint solution (V,Π) with $\Pi \in K \subset C([0,T],\mathbb{R}^n)$ and $V \in C([0,T],\mathbb{R}^n)$. ■

An example for the set of parameters that satisfies the inequality (21) is as follows:

$$\alpha = 0.1, \beta = 0.5, r = 1.0, T = 1.0, c_1 = 0.2,$$
$$c_2 = 0.1, \|V_0\| = 1.0, \|\Sigma\Sigma^\intercal\| = 0.5, M = 1.0,$$

with $\Theta(E) = 0.1E + 0.2I_n$ for any $E \in \mathbb{R}^{n\times n}$.

To establish the existence of MFG solutions, we introduce an additional assumption below.

Theorem 1. Consider $[0,T]$ and $K := \{\Pi \in C([0,T],\mathbb{R}^n) : \|\Pi\|_\infty \le M\}$. Assume (A1) and (A2) hold. If T and M satisfy the relation (21), then the solution to the limit MFG problem specified by (4), (5) and (6) exists and is given by

$$u_i^*(t) = -R^{-1}B^\intercal \Pi_t (x_i(t) - \bar{x}(t)) \tag{23}$$

where $V(\cdot)$ and $\Pi_{(\cdot)}$ are given by the equation pair

$$\dot{V}(t) = (A - BR^{-1}B^\intercal\Pi_t)V(t) + V(t)(A - BR^{-1}B^\intercal\Pi_t)^\intercal + \Sigma\Sigma^\intercal, V(0) = V_0 \tag{24}$$

$$-\dot{\Pi}_t = A^\intercal\Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\intercal\Pi_t + \Theta(V(t)), \Pi_T = Q_T \tag{25}$$

and $\bar{x}(\cdot)$ is given by

$$\dot{\bar{x}}(t) = A\bar{x}(t), \quad \bar{x}(0) = \mu_0. \tag{26}$$

□

PROOF By Proposition 1, the solution pair $(V(\cdot), \Pi_{(\cdot)})$ exists. Hence $\bar{x}(\cdot)$ satisfying (26) exists. Then following Lemma (3) the best response exists and it given by (23), and furthermore it generates the mean field distributions characterized by $(\bar{x}(t), V(t))_{t\in[0,T]}$ that satisfies (26) and (24). There exists at least one Nash equilibrium for the limit MFG problem and the corresponding strategy of a generic agent is given by (23). ■

4.2 Sufficient Condition for Uniqueness

We use the Banach fixed point theorem to establish the existence of a unique solution over a small time horizon. We introduce the assumption below.

Assumption 3. (A4). The function $\Theta : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfies that for all $x_i \in \mathbb{R}^{n\times n}$ with $0 < b_1 \le \|E_i\| \le b_2 < \infty$, $i = 1, 2$, the following hold

$$\|\Theta(E_1) - \Theta(E_2)\| \le c_3(b_1,b_2)\|E_1 - E_2\|,$$

for some $0 < c_3(b_1,b_2) < \infty$ that depend on b_1 and b_2.

Remark 5. Examples of $\Theta(E)$ that satisfy (A3) include: $\Theta(E) = \exp(E)$, $\Theta(E) = \sum_{k=0}^K E^k$, and $\Theta(E) = E^{-1}$. □

Proposition 2. (Uniqueness). Consider $[0,T]$ and $K := \{\Pi \in C([0,T],\mathbb{R}^n) : \|\Pi\|_\infty \leq M\}$. Assume (A1) and (A3) hold. If T and M satisfy the following inequality

$$T(2a + 2\frac{\beta^2}{r}M + c_3(b_1,b_2)2e^{3(\alpha+\frac{\beta^2}{r}M)T}\frac{\beta^2}{r}T(\|V_0\| + T\|\Sigma\Sigma^\intercal\|)) < 1 \tag{27}$$

with $b_1 = e^{-2(\alpha+\frac{\beta^2}{r}M)T}\|V_0\|$, and $b_2 = e^{2(\alpha+\frac{\beta^2}{r}M)T}\big(\|V_0\| + T\|\Sigma\Sigma^\intercal\|)\big)$, then the coupled differential equations (17) and (18) with the coupling constraint (6) have an unique solution pair (V,Π) with $\Pi \in K$ and $V \in C([0,T],\mathbb{R}^n)$. □

PROOF For any $\Pi,\ \tilde{\Pi} \in K \subset C([0,T],\mathbb{R}^n)$, we have

$$\begin{aligned}\|\mathcal{T}(\Pi)(t) - \mathcal{T}(\tilde{\Pi})(t)\| \leq \int_t^T &\|A^\intercal\Pi_q + \Pi_q A \\ &- \Pi_q BR^{-1}B^\intercal\Pi_q - (A^\intercal\tilde{\Pi}_q + \tilde{\Pi}_q A - \tilde{\Pi}_q BR^{-1}B^\intercal\tilde{\Pi}_q) \\ &+ \Theta(I_1(q)) - \Theta(\tilde{I}_1(q))\|dq\end{aligned}$$

where

$$I_1(q) := \Phi(q,0)V_0\Phi^\intercal(q,0) + \int_0^q \Phi(q,s)\Sigma\Sigma^\intercal\Phi^\intercal(q,s)\,ds$$

and $\Phi(\cdot,\cdot)$ given by (20), and

$$\tilde{I}_1(q) := \tilde{\Phi}(q,0)V_0\tilde{\Phi}^\intercal(q,0) + \int_0^q \tilde{\Phi}(q,s)\Sigma\Sigma^\intercal\tilde{\Phi}^\intercal(q,s)\,ds$$

and $\tilde{\Phi}(\cdot,\cdot)$ is the transition matrix for (20) with Π replaced by $\tilde{\Pi}$. By (A3), Lemma 5 and Lemma 7, we obtain that

$$\begin{aligned}&\|\Theta(I_1(q)) - \Theta(\tilde{I}_1(q))\| \leq c_3(b_1,b_2)\|I_1(q) - \tilde{I}_1(q)\| \\ &\leq c_3(b_1,b_2)2e^{3(\alpha+\frac{\beta^2}{r}M)q}\frac{\beta^2}{r}q(\|V_0\| + q\|\Sigma\Sigma^\intercal\|)\|\Pi - \tilde{\Pi}\|_\infty.\end{aligned}$$

Hence

$$\begin{aligned}\|\mathcal{T}(\Pi)(t) - \mathcal{T}(\tilde{\Pi})(t)\| &\leq T(2a + 2\frac{\beta^2}{r}M + c_3(b_1,b_2) \\ &\times 2e^{3(\alpha+\frac{\beta^2}{r}M)T}\frac{\beta^2}{r}T(\|V_0\| + T\|\Sigma\Sigma^\intercal\|))\|\Pi - \tilde{\Pi}\|_\infty.\end{aligned}$$

Then under condition (27), we obtain

$$\|\mathcal{T}(\Pi) - \mathcal{T}(\tilde{\Pi})\|_\infty < \|\Pi - \tilde{\Pi}\|_\infty.$$

If we select M and T such that the inequality (27) holds, then the mapping $\mathcal{T}$ is a contraction under the uniform norm in K over a small interval $[0,T]$. Furthermore, K is complete under the uniform norm since K is a closed subset of the complete metric space $C([0,T],\mathbb{R}^n)$ under the uniform norm (see e.g., (Rudin et al., 1976, p. 54)). Then an application of the Banach fixed point theorem implies that equation (11) has a unique solution $\Pi \in K$. This further implies that the solution pair (V,Π) to the coupled differential equations (17) and (18) is unique with $\Pi \in K$ and $V \in C([0,T],\mathbb{R}^n)$ and hence the proof is complete. ■

An example for the parameters that satisfy the inequality (27) is as follows:

$$\begin{aligned}&T = 0.1, \quad a = 0.5, \quad \beta = 1, \quad r = 1, \quad M = 1, \\ &\alpha = 0.2, \quad c_3 = 0.5, \quad \|V_0\| = 1, \quad \|\Sigma\Sigma^\top\| = 1.\end{aligned}$$

with $\Theta(E) = 0.5E$ for $E \in \mathbb{R}^{n\times n}$.

Based on Proposition 2, we obtain the following result on the uniqueness of MFG solutions.

Theorem 2. Consider $[0,T]$ and $K := \{\Pi \in C([0,T],\mathbb{R}^n) : \|\Pi\|_\infty \leq M\}$. Assume (A1) and (A3) hold. If T and M satisfy the relation (27), then the solution to the limit MFG problem specified by (4), (5) and (6) exists and is unique; furthermore, the MFG solution is given by

$$u_i^*(t) = -R^{-1}B^\intercal\Pi_t(x_i(t) - \bar{x}(t)) \tag{28}$$

where $V(t)$, Π_t and $\bar{x}(t)$ satisfies the same equations in Theorem 1 for all $t \in [0,T]$. □

PROOF By Prop. 2, the solution pair $(V(\cdot),\Pi_{(\cdot)})$ uniquely exists. Hence $\bar{x}(\cdot)$ satisfying (26) uniquely exists. Then following Lemma (3) the best response uniquely exists and it given by (28), and furthermore it uniquely generates the mean field distributions characterized by $(\bar{x}(t),V(t))_{t\in[0,T]}$ that satisfies (26) and (24). Hence the Nash equilibrium for the limit MFG problem is unique and the strategy of a generic agent is given by (28). ■

5. CONCLUSION

This paper investigated a new class of LQG MFG problems where the cost coefficients depend on the covariance matrix of the states and established sufficient conditions for the existence and uniqueness of the MFG solution. Future investigation should include (a) the establishment of more explicit conditions for existence and uniqueness of solutions, (b) the consideration of cases where the cost coefficients depend on both the mean and the covariance matrix, and (c) the establishment of ε-Nash properties of the MFG solution.

Appendix A. LEMMAS USED IN THE PROOFS

Lemma 4. If $\|\Pi_{(\cdot)}\|_\infty \leq M$, then the following properties hold for all $s,t \in [0,T]$ with $s \leq t$:

(a) $\|A^\intercal\Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\intercal\Pi_t\| \leq 2\alpha M + \frac{\beta^2}{r}M^2$,

(b) $\|A - BR^{-1}B^\intercal\Pi_t\| \leq \alpha + \frac{\beta^2}{r}M$,

(c) $\|\Phi(t,s)\| \leq e^{(\alpha+\frac{\beta^2}{r}M)(t-s)}$,

(d) $\|\Phi(t,s)\| \geq e^{-(\alpha+\frac{\beta^2}{r}M)(t-s)}$,

with $\Phi(\cdot,\cdot)$ given by (20). □

Lemma 5. Let $\|A\| = \alpha$, $\|B\| = \beta$, $\|R\| = r > 0$. If $\|\Pi_{(\cdot)}\|_\infty \leq M$, then the following hold:

$$\begin{aligned}\big\|I_1(q)\big\| &\leq e^{2(\alpha+\frac{\beta^2}{r}M)q}\|V_0\| + \int_0^q e^{2(\alpha+\frac{\beta^2}{r}M)(q-s)}\|\Sigma\Sigma^\intercal\|ds \\ &\leq e^{2(\alpha+\frac{\beta^2}{r}M)T}\Big(\|V_0\| + T\|\Sigma\Sigma^\intercal\|)\Big), \\ \big\|I_1(q)\big\| &\geq e^{-2(\alpha+\frac{\beta^2}{r}M)q}\|V_0\| \geq e^{-2(\alpha+\frac{\beta^2}{r}M)T}\|V_0\|,\end{aligned}$$

with

$$I_1(q) := \Phi(q,0)V_0\Phi^\intercal(q,0) + \int_0^q \Phi(q,s)\Sigma\Sigma^\intercal\Phi^\intercal(q,s)\,ds$$

and $\Phi(\cdot,\cdot)$ given by (20). □

Lemma 6. (Equicontinuity). Let $\mathcal{T}$ be given by (22) and $K := \{\Pi \in C([0,T],\mathbb{R}^n) : \|\Pi\|_\infty \leq M\}$ for some $M > 0$. Assume (A1)-(A2) hold. Then the set $\{\mathcal{T}(\Pi) : \Pi \in K\}$ is equicontinuous. □

PROOF To show the equicontinuity (Rudin et al., 1976, Def. 7.22) of $\{\mathcal{T}(\Pi) : \Pi \in K\}$ with $\mathcal{T}$ given by (22), we need to prove that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any $t_1,t_2 \in [0,T]$ with $|t_1 - t_2| < \delta$, the following inequality holds $\|\mathcal{T}(\Pi)(t_1) - \mathcal{T}(\Pi)(t_2)\| < \epsilon$ for all $\Pi \in K$. Consider the difference:

$$\|\mathcal{T}(\Pi)(t_1) - \mathcal{T}(\Pi)(t_2)\| \le \int_{t_1}^{t_2} \|I_2(q)\|\, dq$$

where $I_2(q)$ denotes the integrand for $\mathcal{T}(\Pi)(t)$ given by

$$I_2(q) = A^\intercal \Pi_q + \Pi_q A - \Pi_q BR^{-1}B^\intercal \Pi_q + \Theta\big(I_1(q)\big)$$

with

$$I_1(q) = \Phi(q,0)V_0\Phi^\intercal(q,0) + \int_0^q \Phi(q,s)\Sigma\Sigma^\intercal\Phi^\intercal(q,s)ds.$$

Since $\|\Pi\|_\infty \le M$ for a given $M > 0$, the terms inside the integrand $I(q)$ are uniformly bounded: firstly,

$$\|A^\intercal \Pi_q + \Pi_q A - \Pi_q BR^{-1}B^\intercal \Pi_q\| \le 2\alpha M + \frac{\beta^2}{r}M^2;$$

secondly, from Lemma 5 and (A2), we have

$$\|\Theta(I_1(q))\| \le c_1 + c_2(b_1,b_2)\|I_1(q)\|.$$

where $b_1 = e^{-2(\alpha+\frac{\beta^2}{r}M)T}\|V_0\|$ and $b_2 = e^{2(\alpha+\frac{\beta^2}{r}M)T}\big(\|V_0\| + T\|\Sigma\Sigma^\intercal\|\big)$. Let the uniform bound be denoted by

$$\begin{aligned} C := (2\alpha M + \frac{\beta^2}{r}M^2 + c_1) \\ + c_2(b_1,b_2)e^{2(\alpha+\frac{\beta^2}{r}M)T}\Big(\|V_0\| + T\|\Sigma\Sigma^\intercal\|\Big). \end{aligned}$$

Then we have

$$\begin{aligned} \|\mathcal{T}(\Pi)(t_1) - \mathcal{T}(\Pi)(t_2)\| &\le \int_{t_1}^{t_2} \|I(s)\|\, ds \\ &\le \int_{t_1}^{t_2} C\, ds = C|t_2 - t_1|. \end{aligned}$$

Hence, if we choose $\delta = \frac{\varepsilon}{C}$, then $\|\mathcal{T}(\Pi)(t_1) - \mathcal{T}(\Pi)(t_2)\| < \varepsilon$ for all $\Pi \in K$ whenever $|t_1 - t_2| < \delta$. That is, the set $\{\mathcal{T}(\Pi) : \Pi \in K\}$ is equicontinuous. ■

Lemma 7. If $\|\Pi_{(\cdot)}\|_\infty \le M$ and $\|\tilde{\Pi}_{(\cdot)}\|_\infty \le M$, then for all $q \in [0,T]$

$$\begin{aligned} &\|I_1(q) - \tilde{I}_1(q)\| \\ &\quad \le 2e^{3(\alpha+\frac{\beta^2}{r}M)q}\frac{\beta^2}{r}q(\|V_0\| + q\|\Sigma\Sigma^\intercal\|)\|\Pi - \tilde{\Pi}\|_\infty \end{aligned}$$

where

$$I_1(q) := \Phi(q,0)V_0\Phi^\intercal(q,0) + \int_0^q \Phi(q,s)\Sigma\Sigma^\intercal\Phi^\intercal(q,s)\, ds$$

with $\Phi(\cdot,\cdot)$ given by (20),

$$\tilde{I}_1(q) := \tilde{\Phi}(q,0)V_0\tilde{\Phi}^\intercal(q,0) + \int_0^q \tilde{\Phi}(q,s)\Sigma\Sigma^\intercal\tilde{\Phi}^\intercal(q,s)\, ds$$

and $\tilde{\Phi}(\cdot,\cdot)$ is the transition matrix for (20) with Π replaced by $\tilde{\Pi}$. □

PROOF Let $\Delta(t,s) := \Phi(t,s) - \tilde{\Phi}(t,s)$. It satisfies

$$\begin{aligned} &\frac{\partial}{\partial t}\Delta(t,s) \\ &= (A - BR^{-1}B^\intercal\Pi_t)\Phi(t,s) - (A - BR^{-1}B^\intercal\tilde{\Pi}_t)\tilde{\Phi}(t,s) \\ &= (A - BR^{-1}B^\intercal\Pi_t)\Delta(t,s) - BR^{-1}B^\intercal(\Pi_t - \tilde{\Pi}_t)\tilde{\Phi}(t,s) \end{aligned}$$

with $\Delta(s,s) = 0$. Hence

$$\Delta(t,s) = \int_s^t \Phi(t,\tau)BR^{-1}B^\intercal(\tilde{\Pi}_\tau - \Pi_\tau)\tilde{\Phi}(\tau,s)\, d\tau.$$

Then by Lemma 4

$$\begin{aligned} &\|\Delta(t,s)\| \\ &\le \int_s^t \|\Phi(t,\tau)\| \cdot \|BR^{-1}B^\intercal\| \cdot \|\Pi_\tau - \tilde{\Pi}_\tau\| \cdot \|\tilde{\Phi}(\tau,s)\|\, d\tau \\ &\le e^{2(\alpha+\frac{\beta^2}{r}M)(t-s)}\frac{\beta^2}{r}\int_s^t \|\Pi_\tau - \tilde{\Pi}_\tau\|\, d\tau. \end{aligned}$$

Thus

$$\begin{aligned} &\big\|\Phi(q,0)V_0\Phi^\intercal(q,0) - \tilde{\Phi}(q,0)V_0\tilde{\Phi}^\intercal(q,0)\big\| \\ &\quad \le \|\Phi(q,0) - \tilde{\Phi}(q,0)\| \cdot \|V_0\| \cdot \|\Phi^\intercal(q,0)\| \\ &\qquad + \|\tilde{\Phi}(q,0)\| \cdot \|V_0\| \cdot \|\Phi(q,0) - \tilde{\Phi}(q,0)\| \\ &\quad \le 2\|\Delta(q,0)\|\|V_0\|e^{(\alpha+\frac{\beta^2}{r}M)q} \\ &\quad \le 2e^{3(\alpha+\frac{\beta^2}{r}M)q}\|V_0\|\frac{\beta^2}{r}\int_0^q \|\Pi_\tau - \tilde{\Pi}_\tau\|\, d\tau. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} &\|\Phi(q,s)\Sigma\Sigma^\intercal\Phi^\intercal(q,s) - \int_0^q \tilde{\Phi}(q,s)\Sigma\Sigma^\intercal\tilde{\Phi}^\intercal(q,s)\| \\ &\quad \le 2e^{3(\alpha+\frac{\beta^2}{r}M)(q-s)}\|\Sigma\Sigma^\intercal\|\frac{\beta^2}{r}\int_s^q \|\Pi_\tau - \tilde{\Pi}_\tau\|\, d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|I_1(q) - \tilde{I}_1(q)\| \\ &\le \|\Phi(q,0)V_0\Phi^\intercal(q,0) - \tilde{\Phi}(q,0)V_0\tilde{\Phi}^\intercal(q,0)\| \\ &+ \int_0^q \|\Phi(q,s)\Sigma\Sigma^\intercal\Phi^\intercal(q,s) - \int_0^q \tilde{\Phi}(q,s)\Sigma\Sigma^\intercal\tilde{\Phi}^\intercal(q,s)\|ds \\ &\le 2e^{3(\alpha+\frac{\beta^2}{r}M)q}\|V_0\|\frac{\beta^2}{r}\int_0^q \|\Pi_\tau - \tilde{\Pi}_\tau\|\, d\tau \\ &+ \int_0^q 2e^{3(\alpha+\frac{\beta^2}{r}M)(q-s)}\|\Sigma\Sigma^\intercal\|\frac{\beta^2}{r}\int_s^q \|\Pi_\tau - \tilde{\Pi}_\tau\|\, d\tau dq \\ &\le 2e^{3(\alpha+\frac{\beta^2}{r}M)q}\|V_0\|\frac{\beta^2}{r}q\|\Pi - \tilde{\Pi}\|_\infty \\ &+ q2e^{3(\alpha+\frac{\beta^2}{r}M)q}\|\Sigma\Sigma^\intercal\|\frac{\beta^2}{r}q\|\Pi - \tilde{\Pi}\|_\infty \\ &= 2e^{3(\alpha+\frac{\beta^2}{r}M)q}\frac{\beta^2}{r}q(\|V_0\| + q\|\Sigma\Sigma^\intercal\|)\|\Pi - \tilde{\Pi}\|_\infty. \end{aligned}$$

■

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