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Graphon-LQR Control of Arbitrary Size Networks of Linear Systems

Shuang Gao and Peter E. Caines

***Abstract*— To achieve control objectives for extremely complex and very large scale networks using standard methods is essentially intractable. In this paper, we exploit our previously proposed graphon control methodology to approximately regulate complex network systems by the use of graphon theory and the theory of infinite dimensional systems. Conditions on the exact controllability and the approximate controllability on graphon dynamical systems are investigated. Approximation schemes to approximately regulate large network systems with linear quadratic cost are developed. The convergence properties of the approximation schemes are proved. Finally, two simulations of the application of graphon-LQR control to complex networks are presented.**

***Index Terms*— Graphon control, large networks, complex networks, graphons, infinite dimensional systems**

I. Introduction

Complex network systems such as biological, gene, brain, citation, electric and social networks, are ubiquitous, and the study of large scale networks has been the focus of much research over the past 15 years. In particular, researchers have been studying networks of interacting dynamical systems to learn which collective behaviours may emerge from system interactions over a complex network ([1]). Furthermore, in addition to the structural properties of networks, system theoretic notions such as controllability, observability, consensus dynamics and synchronization have been widely applied ([2]–[10]). In fact, to achieve general control objectives for extremely complex and very large scale networks (henceforth, complex networks) using such standard methods is usually an intractable task. In response to this, we proposed what we term as graphon control theoretic methods in [11]. In that work, the minimum energy state to state control problem is analysed. In this work, we further develop the graphon control theoretic methods to solve the regulation problem on complex network systems with linear quadratic costs. In addition, we investigate conditions on the exact controllability and the approximate controllability on graphon dynamical systems.

Consider the problem of applying linear quadratic regulation (LQR) to each member of a sequence $\tilde{S}$ of networks. The proposed control strategy consists of the following steps: (1) Identify the graphon limit of the sequence of networks as the number of nodes goes to infinity. (2) Solve the corresponding LQR problem for the limit system by solving the limit system Riccati equation. (3) Approximate the Riccati equation solution for the limit system so as to generate approximated control laws for finite network systems. Alternatively, directly approximate the control input generated for the limit system and apply the result to the networks of systems along the sequence $\tilde{S}$.

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II. Preliminaries

A. Graphs, Adjacency Matrices and Pixel Pictures

The underlying structure of a network can be described by a graph $G = (V, E)$ specified by a vertex set V and an edge set E which represents the connections between vertices. An equivalent representation of a graph $G = (V, E)$ by a matrix called an *adjacency matrix* is defined to be the square $|V| \times |V|$ matrix A such that an element A_{ij} is one when there is an edge from vertex i to vertex j, and zero otherwise. If the graph is a weighted graph where edges are associated with weights, then the adjacency matrix has corresponding weighted elements.

Another representation of the adjacency matrix is given by a pixel diagram where the 0s are replaced by white squares and the 1s by black squares. The whole pixel diagram is presented in a unit square, so the square elements have sides of length $\frac{1}{n}$, where n is the number of vertices.

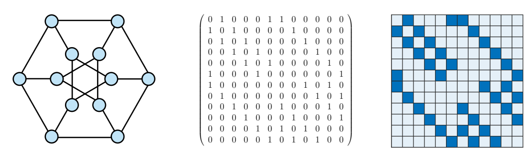


Fig. 1. Dürer Graph, Adjacency Matrix, Pixel Diagram

B. Graphon

Graphon theory was introduced and developed in recent years by L. Lovász, B. Szegedy, C. Borgs, J. T. Chayes, V. T. Sós, and K. Vesztergombi among others in [12]–[16]. This work draws on graph theory, measure theory, probability, and functional analysis. In the literature(see e.g. [16]), a meaningful convergence with respect to the *cut metric* is defined for sequences of dense and finite graphs. Graphons are then the limit objects of converging graph sequences. This concept is illustrated by a sequence of half graphs ([16]) represented by a sequence of pixel diagrams on the unit square converging to its limit in Fig. 2.

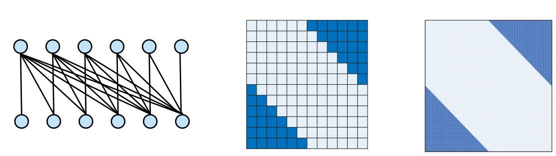


Fig. 2. Graph Sequence Converging to Its Limit

The set of finite graphs endowed with the cut metric gives rise to a metric space, and the completion of this space is the *space of graphons*. Graphons are represented by bounded symmetric Lebesgue measurable functions $W : [0,1]^2 \rightarrow [0,1]$, which can be interpreted as weighted graphs on the vertex set $[0,1]$. We note that in some papers, for instance [17], the word "graphon" refers to symmetric, integrable functions from $[0,1]$ to R. In this paper, unless stated otherwise, the term "graphon" is used to refer to functions $\mathbf{W_1} : [0,1]^2 \rightarrow [-1,1]$ and $\mathbf{\tilde{G}_1^{sp}}$ denotes the space of graphons. Let $\mathbf{\tilde{G}_0^{sp}}$ represent the space of all graphons satisfying $\mathbf{W_0} : [0,1]^2 \rightarrow [0,1]$; let $\mathbf{\tilde{G}^{sp}}$ denote the space of all symmetric measurable functions $\mathbf{W} : [0,1]^2 \rightarrow R$.

The cut norm of a graphon is then defined as

$$\|\mathbf{W}\|_\square = \sup_{M,T\subset[0,1]} |\int_{M\times T} \mathbf{W}(x,y)dxdy| \tag{1}$$

with the supremum taking over all measurable subsets M and T of $[0,1]$. The inequalities between the different norms on a graphon $\mathbf{W}$ are

$$\|\mathbf{W}\|_\square \leq \|\mathbf{W}\|_1 \leq \|\mathbf{W}\|_2 \leq \|\mathbf{W}\|_\infty \leq 1. \tag{2}$$

Denote the set of measure preserving bijections from $[0,1]$ to $[0,1]$ by $S_{[0,1]}$. The *cut metric* between two graphons $\mathbf{V}$ and $\mathbf{W}$ is then given by

$$d_\square(\mathbf{W},\mathbf{V}) = \inf_{\phi\in S_{[0,1]}} \|\mathbf{W}^\phi - \mathbf{V}\|_\square, \tag{3}$$

where $\mathbf{W}^\phi(x,y) = \mathbf{W}(\phi(x),\phi(y))$. We see that the cut metric $d_\square(\cdot,\cdot)$ is given by measuring the maximum discrepancy between the integrals of two graphons over measurable subsets of $[0,1]$, then minimizing the maximum discrepancy over all possible measure preserving bijections.

Since the cut metric of two different graphons can be 0, strictly speaking it is not a metric. See [14], [18] for various characterizations of when the cut distance is 0. By identifying functions $\mathbf{V}$ and $\mathbf{W}$ for which $d_\square(\mathbf{V},\mathbf{W}) = 0$, we can construct the metric space $\mathbf{G_1^{sp}}$ which denotes the image of $\mathbf{\tilde{G}_1^{sp}}$ under this identification. Similarly we construct $\mathbf{G_0^{sp}}$ from $\mathbf{\tilde{G}_0^{sp}}$ and $\mathbf{G^{sp}}$ from $\mathbf{\tilde{G}^{sp}}$.

We define the L^2 *metric* for any graphons $\mathbf{W}$ and $\mathbf{V}$ as

$$\begin{aligned} d_{L^2}(\mathbf{W},\mathbf{V}) &= \inf_{\phi\in S_{[0,1]}} \|\mathbf{W}^\phi - \mathbf{V}\|_2 \\ &= \inf_{\phi\in S_{[0,1]}} \left(\int_{[0,1]^2} |\mathbf{W}^\phi(x,y) - \mathbf{V}(x,y)|^2 dxdy\right)^{\frac{1}{2}}. \end{aligned}$$

Then we can prove that for any two graphons $\mathbf{W}$ and $\mathbf{V}$

$$d_\square(\mathbf{W},\mathbf{V}) \leq d_{L^2}(\mathbf{W},\mathbf{V}). \tag{4}$$

C. Compactness of the Graphon Space

Theorem 1 ([16]). The space $(\mathbf{G_0^{sp}}, d_\square)$ is compact.

This remains valid if $\mathbf{G_0^{sp}}$ is replaced by any uniformly bounded subset of $\mathbf{G^{sp}}$ closed in the cut metric [16].

Theorem 2 ([16]). The space $(\mathbf{G_1^{sp}}, d_\square)$ is compact.

Sets in $\mathbf{G_1^{sp}}$ compact with respect to the L^2 metric are compact with respect to the cut metric. It follows immediately from (4) and Theorem 2 (or Theorem 1), if a graphon sequence is Cauchy in the L^2 metric then it is also a Cauchy sequence in the cut metric and under both metrics, the limits are identical in $\mathbf{G_1^{sp}}$ (or $\mathbf{G_0^{sp}}$).

D. Step Functions in the Graphon Space

Graphons generalize weighted graphs in the following sense (see [16]). A function $\mathbf{W} \in \mathbf{G_1^{sp}}$ is called a *step function* if there is a partition $Q = \{Q_1, ..., Q_k\}$ of $[0,1]$ into measurable sets such that $\mathbf{W}$ is constant on every product set $Q_i \times Q_j$. The sets Q_i are the *steps* of $\mathbf{W}$. For every weighted graph G (on node set $V(G)$), a step function $\mathbf{S_G} \in \mathbf{G_1^{sp}}$ is given as follows: partition $[0,1]$ into n measurable sets $Q_1, \cdots, Q_n$ of measure $\mu(Q_i) = \frac{\alpha_i}{\alpha_G}$, then for $x \in Q_i$ and $y \in Q_j$, we let $\mathbf{S_G}(x,y) = \beta_{ij}(G)$, where α_i denotes the node weight of i^{th} node, $\alpha(G) = \sum_i \alpha_i$ and $\beta_{ij}(G)$ denotes the weight of the edge from node i to node j (i.e., β_{ij} is the ij^{th} entry in the adjacency matrix of G). Evidently the function $\mathbf{S_G}$ depends on the labelling of the nodes of G. We define the *uniform partition* $P^N = \{P_1, P_2, ..., P_N\}$ of $[0,1]$ by setting $P_k = [\frac{k-1}{N}, \frac{k}{N}), k \in \{1, N-1\}$ and $P_N = [\frac{N-1}{N}, 1]$. Then $\mu(P_i) = \frac{1}{N}$, $i \in \{1,2,...,N\}$. Under the uniform partition, the step functions can be represented by the pixel diagram on the unit square.

E. Graphons as Operators

Following [16], a graphon $\mathbf{W} \in \mathbf{G_1^{sp}}$ can be interpreted as an operator $\mathbf{W} : L^2[0,1] \rightarrow L^2[0,1]$. The operation on $\mathbf{v} \in L^2[0,1]$ is defined as follows:

$$[\mathbf{W}\mathbf{v}](x) = \int_0^1 \mathbf{W}(x,\alpha)\mathbf{v}(\alpha)d\alpha. \tag{5}$$

The operator product is then defined by

$$[\mathbf{U}\mathbf{W}](x,y) = \int_0^1 \mathbf{U}(x,z)\mathbf{W}(z,y)dz, \tag{6}$$

where $\mathbf{U},\mathbf{W} \in \mathbf{G_1^{sp}}$. Note that if $\mathbf{U} \in \mathbf{G_1^{sp}}$ and $\mathbf{W} \in \mathbf{G_1^{sp}}$, then $\mathbf{U}\mathbf{W} \in \mathbf{G_1^{sp}}$, since for all $x,y \in [0,1]$

$$\begin{aligned} |[\mathbf{U}\mathbf{W}](x,y)| &= |\int_0^1 \mathbf{U}(x,z)\mathbf{W}(z,y)dz| \\ &\leq \int_0^1 |\mathbf{U}(x,z)\mathbf{W}(z,y)|dz \leq 1. \end{aligned} \tag{7}$$

Consequently, the power $\mathbf{W}^n$ of an operator $\mathbf{W} \in \mathbf{G_1^{sp}}$ is defined as

$$\mathbf{W}^n(x,y) = \int_{[0,1]^n} \mathbf{W}(x,\alpha_1)\cdots\mathbf{W}(\alpha_{n-1},y)d\alpha_1\cdots d\alpha_{n-1}$$

with $\mathbf{W}^n \in \mathbf{G_1^{sp}}$ $(n \geq 1)$. $\mathbf{W^0}$ is formally defined as the identity operator on functions in $L^2[0,1]$, but we note that $\mathbf{W^0}$ is not a graphon.

F. The Graphon Unitary Operator Algebra

We have an operator algebra $\mathcal{G}_{\mathcal{A}}$ over the field R (see [11]) acting on elements of $L^2[0,1]$ as given by equation (5). By adjoining the identity element I to the algebra $\mathcal{G}_{\mathcal{A}}$ we obtain a unitary algebra $\mathcal{G}_{\mathcal{AI}}$. The identity element I is defined as follows: for any $\mathbf{W} \in L^2[0,1]^2$

$$[I\mathbf{W}](x,y) = \int_0^1 \mathbf{W}(z,y)\delta(x,z)dz = \mathbf{W}(x,y), \qquad (8)$$

where $\delta(\cdot,z)dz$ is the measure satisfying $\int_0^1 u(z)\delta(x,z)dz = u(x)$ for all $u \in L^2[0,1]$, and in particular $\int_0^1 \delta(x,z)dz = 1$. The graphon unitary operator algebra $\mathcal{G}_{\mathcal{AI}}$ will be used in the definition of the controllability Gramian and the input operator. More specifically, we use the subset $\mathcal{G}^1_{\mathcal{AI}} = \{\mathcal{G}^1_{\mathcal{A}}, I\}$ where $\mathcal{G}^1_{\mathcal{A}}$ is the set in $\mathcal{G}_{\mathcal{A}}$ that corresponds to $\widetilde{\mathbf{G}}^{\mathbf{sp}}_{\mathbf{1}}$.

G. Graphon Differential Equations

Let X be a Banach space. A *linear operator* $A : D(A) \subset X \to X$ is *closed* if $\{(x, Ax) : x \in D(A)\}$ is closed in the product space $X \times X$(see [19]). $\mathcal{L}(X)$ denotes the Banach algebra of all linear continuous mappings $T : X \to X$. $L^p(a,b;X)$ denotes the Banach space of equivalent classes of strongly measurable (in the Böchner sense) mappings $[a,b] \to X$ that are p-integrable, $1 \le p < \infty$, with norm $\|f\|_{L^p(a,b;X)} = \left[\int_a^b |f(s)|^p ds\right]^{\frac{1}{p}}$. Let $\mathbf{A} : [0,1]^2 \to [-1,1]$ be a graphon and hence a bounded and closed linear operator from $L^2[0,1]$ to $L^2[0,1]$. Following [20], $\mathbf{A}$ is the infinitesimal generator of the uniformly (hence strongly) continuous semigroup $S_{\mathbf{A}}(t) := e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k \mathbf{A}^k}{k!}$. Therefore, the initial value problem of the graphon differential equation

$$\dot{\mathbf{y}}_{\mathbf{t}} = \mathbf{A}\mathbf{y}_{\mathbf{t}}, \quad \mathbf{y}_{\mathbf{0}} \in L^2[0,1] \qquad (9)$$

has a solution given by $\mathbf{y}_{\mathbf{t}} = e^{\mathbf{A}t}\mathbf{y}_{\mathbf{0}}$.

Theorem 3 ([11]). Let $\{\mathbf{A}_{\mathbf{N}}\}_{N=1}^{\infty}$ be a sequence of graphons such that $\mathbf{A}_{\mathbf{N}} \to \mathbf{A}_*$ as $N \to \infty$ in the L^2 metric. Then for all $\mathbf{x} \in L^2[0,1]$, $e^{\mathbf{A}_{\mathbf{N}}t}\mathbf{x} \to e^{\mathbf{A}_*t}\mathbf{x}$ as $N \to \infty$ in the L^2 metric where the convergence is pointwise in time and uniform on any time interval $[0,T]$.

III. Network Systems and Their Limit Systems

A. Scaled Network Systems with Node Averaging Dynamics

Consider an interlinked network S^N of linear (symmetric) dynamical subsystems $\{S_i^N; 1 \le i \le N\}$, each with an n dimensional state space. The subsystem S_i^N at the node V_i in the network $G_N(V,E)$ has interactions with $S_j^N, 1 \le j \le N$, specified as below:

$$S_i^N: \quad \begin{aligned} \dot{x}_t^i &= \frac{1}{nN}\sum_{j=1}^{N} A_{ij}x_t^j + \frac{1}{nN}\sum_{j=1}^{N} B_{ij}u_t^j, \\ & x_t^i, u_t^i \in R^n, i \in \{1,...,N\}, \end{aligned}$$

with $A_N = [A_{ij}], B_N = [B_{ij}] \in R^{nN \times nN}$, the (symmetric) block-wise adjacency matrices of $G_N(V,E)$ and of the input graph, where $A_{ij} = [0]$ if S_i^N has no connection to S_j^N and similarity for B_{ij}. Then the (symmetric) linear dynamics for the network system $S^N(A_N, B_N, G_N)$ can be represented by

$$S^N: \quad \begin{aligned} &\dot{x}_t = A_N \circ x_t + B_N \circ u_t, \\ &x_t, u_t \in R^{nN}, \quad A_N, B_N \in R^{nN \times nN}, \end{aligned} \qquad (10)$$

where $\circ$ denotes the so called averaging operator given by $A_N \circ x = \frac{1}{(nN)} A_N x$. Let $\mathcal{S} = \times_{N=1}^{\infty} \mathcal{S}^{\mathcal{N}}$ where $\mathcal{S}^{\mathcal{N}} = \cup_{A_N,B_N,G_N} S^N(A_N,B_N,G_N)$. For simplicity, we require the elements of A_N and B_N to be in $[-1,1]$ for each N (note that in general A_N and B_N have elements that are bounded real numbers for which case we would achieve similar results). In addition, we note that if we take the supremum norm on vectors in R^{nN}, i.e. $\|x\|_\infty = \sup_i |x_i|$, and the corresponding $\circ$ operator norm of A, i.e. $\|A\|_{op} = \sup_{\|x\|_\infty \neq 0} \frac{\|A \circ x\|_\infty}{\|x\|_\infty}$, then $\|A\|_{op} \le 1$.

B. Network Systems with Node Averaging Dynamics Described by Step Functions in the Graphon Space

Let $\{(A_N; B_N)\}_{N=1}^{\infty} \in \mathcal{S}$ be a sequence of systems with the node averaging dynamics each of which is described according to (10). Let $|A_{Nij}| \le 1$ and $|B_{Nij}| \le 1$ for all $i,j \in \{1,...,nN\}$. Let $\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}, \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]} \in \mathbf{G}^{\mathbf{sp}}_{\mathbf{1}}$ be the step functions corresponding one-to-one to A_N and B_N; these are specified using the uniform partition P^{nN} of $[0,1]$ by the following *matrix to step function mapping* M_G: for all $i,j \in \{1,2,...,nN\}$,

$$\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}(x,y) := A_{Nij}, \quad \forall (x,y) \in P_i \times P_j, \qquad (11)$$

and similar for $\mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]}$.

Define a *piece-wise constant function* on R to be any function of the form $\sum_{k=1}^{l} \alpha_k \psi_{I_k}$ where $\alpha_1,...,\alpha_l$ are complex numbers and each I_k is a bounded interval (open, closed, or half-open). Let $L^2_{pwc}[0,1]$ denote the space of piece-wise constant $L^2[0,1]$ functions under the uniform partition P^{nN}.

Let $\mathbf{u}_{\mathbf{t}}^{\mathbf{s}} \in L^2_{pwc}[0,1]$ correspond one-to-one to $u_t \in R^{nN}$ via the following *vector to step function mapping* also denoted by M_G: for all $i \in \{1,...,nN\}$,

$$\mathbf{u}_{\mathbf{t}}^{\mathbf{s}}(\alpha) := u_t(i), \quad \forall \alpha \in P_i, \qquad (12)$$

and $\mathbf{x}_{\mathbf{t}}^{\mathbf{s}} \in L^2_{pwc}[0,1]$ similarly correspond one-to-one to $x_t \in R^{nN}$.

Lemma 1 ([11]). The trajectories of the system in (10) correspond one-to-one under the mapping M_G to the trajectories of the system

$$\begin{aligned} &\dot{\mathbf{x}}_{\mathbf{t}}^{\mathbf{s}} = \mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}\mathbf{x}_{\mathbf{t}}^{\mathbf{s}} + \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]}\mathbf{u}_{\mathbf{t}}^{\mathbf{s}}, \\ &\mathbf{x}_{\mathbf{t}}^{\mathbf{s}}, \mathbf{u}_{\mathbf{t}}^{\mathbf{s}} \in L^2_{pwc}[0,1], \mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}, \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]} \in \mathbf{G}^{\mathbf{sp}}_{\mathbf{1}} \subset \mathcal{G}^1_{\mathcal{AI}} \end{aligned} \qquad (13)$$

with graphon operations defined according to (5).

C. Limits of Sequences of Network Systems

Now the sequence of network systems with the node averaging dynamics can be described by the sequence of step function operators as $\{(\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}; \mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]})\}_{N=1}^{\infty}$. Let the graphon sequences $\{\mathbf{A}_{\mathbf{s}}^{[\mathbf{N}]}\}$ and $\{\mathbf{B}_{\mathbf{s}}^{[\mathbf{N}]}\}$ be Cauchy sequences of step functions in $L^2[0,1]^2$ (under the same measure preserving transformation). Due to the completeness of $L^2[0,1]^2$, the respective graphon limits $\mathbf{A}$ and $\mathbf{B}$ exist and these will then necessarily be the limits in the cut metric (see [16]).

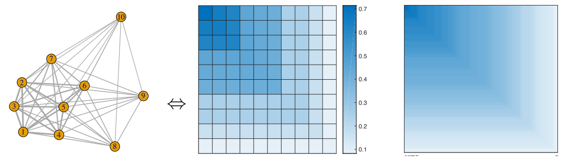


Fig. 3. A Weighted Graph from a Sequence Converging to the Limit Graphon $W(x,y) = 1 - \max(x,y), 0 \le x,y \le 1$ with x,y measured from the top left

IV. The Limit Graphon System and Its Properties

A. Infinite Dimensional Graphon Systems

We follow [19] and specialize the Hilbert space of states H and the Hilbert space of controls U appearing there to the space $L^2(R; L^2[0,1])$. We formulate an infinite dimensional linear system as follows:

$$LS^\infty: \quad \dot{\mathbf{x}}_\mathbf{t} = \mathbf{A}\mathbf{x}_\mathbf{t} + \mathbf{B}\mathbf{u}_\mathbf{t}, \quad \mathbf{x_0} \in L^2[0,1], \tag{14}$$

where $\mathbf{A} \in \mathbf{G_1^{sp}}$, $\mathbf{B} \in \mathcal{G}^1_{\mathcal{AI}}$, and hence bounded operators on $L^2[0,1]$, $\mathbf{x_t} \in L^2[0,1]$ is the system state at time t and $\mathbf{u_t} \in L^2[0,1]$ is the control input at time t.

B. Uniqueness of the Solution

A solution $\mathbf{x}_{(\cdot)} \in L^2(R; L^2[0,1])$ is a *(mild) solution* of (14) if $\mathbf{x_t} = e^{(t-a)\mathbf{A}}\mathbf{x_a} + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}\mathbf{u_s}ds$ for all a and t in R such that $a \le t$ (see [19]). Following [19] the assumptions on the operators $\mathbf{A}$ and $\mathbf{B}$ are

$$(H1) \quad \begin{cases} \text{(i)} \quad \mathbf{A} \text{ generates a strongly continuous} \\ \qquad \text{semigroup } e^{t\mathbf{A}} \text{ on } L^2[0,1], \\ \text{(ii)} \quad \mathbf{B} \in \mathcal{L}(L^2[0,1]; L^2[0,1]), \end{cases}$$

where the Hilbert space U(control space) in the present case is $L^2[0,1]$. Under assumption $(H1)$, the system (14) has a unique solution $\mathbf{x} \in C([0,T]; L^2[0,1])$ for any $\mathbf{x_0} \in L^2[0,1]$ and any $\mathbf{u} \in L^2([0,T]; L^2[0,1])$.

Theorem 4 ([11]). The graphon system LS^∞ in Eq. (14) has a unique solution $\mathbf{x} \in C([0,T]; L^2[0,1])$ for any $\mathbf{x_0} \in L^2[0,1]$ and any $\mathbf{u} \in L^2([0,T]; L^2[0,1])$.

C. Controllability

A system $(\mathbf{A};\mathbf{B})$ is *exactly controllable* on $[0,T]$ if for any initial state $\mathbf{x}_0 \in L^2[0,1]$ and any target state $\mathbf{x}_f \in L^2[0,1]$, there exists a control $\mathbf{u} \in L^2(0,T;U)$ driving the system from $\mathbf{x}_0$ to $\mathbf{x}_f$, i.e. $\mathbf{x}_T = \mathbf{x}_f$ with $\mathbf{x}_T = e^{\mathbf{A}T}\mathbf{x_0} + \int_0^T e^{\mathbf{A}(T-t)}\mathbf{B}\mathbf{u_t}dt$.

A system $(\mathbf{A};\mathbf{B})$ is *approximately controllable* on $[0,T]$ if for any initial state $\mathbf{x}_0 \in L^2[0,1]$, any target state $\mathbf{x}_f \in L^2[0,1]$ and any $\varepsilon > 0$, there exists a control $\mathbf{u} \in L^2(0,T;U)$ driving the system from $\mathbf{x}_0$ to points in the state space within a ε-distance from $\mathbf{x}_f$, i.e. $\|\mathbf{x}_T - \mathbf{x}_f\|_2 \le \varepsilon$.

The *controllability Gramian operator* $\mathbf{W_t} : L^2[0,1] \to L^2[0,1]$ is defined as

$$\mathbf{W_t} := \int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t-s)}ds, \quad t > 0.$$

A necessary and sufficient condition for exact controllability on $[0,T]$ is the uniform positive definiteness of $\mathbf{W}_T$:

$$(\mathbf{W}_T h, h) \ge c_T \|h\|^2$$

for all $h \in L^2[0,1]$, where $c_T > 0$ and $\|\cdot\|$ is the $L^2[0,1]$ norm (see [19], [21]). The positive definiteness of the controllability Gramian operator $\mathbf{W}_T$ as a kernel is equivalent to the approximate controllability of the corresponding system (see [19], [21]).

Theorem 5 ([22]). Let $\mathbf{A}$ be a graphon in $\mathbf{G_1^{sp}}$ and let $\mathbf{B}$ be a bounded linear $L^2[0,1]$ operator. Then $(\mathbf{A};\mathbf{B})$ exactly controllable implies $\mathbf{B}$ is a non-compact operator.

Proposition 1 ([22]). Let $\mathbf{A}$ be a graphon in $\mathbf{G_1^{sp}}$ and let $\mathbf{B}$ be a bounded linear $L^2[0,1]$ operator such that all eigenvalues of $\mathbf{B}\mathbf{B}^T$ are lower bounded by a positive constant $c > 0$. Then $\mathbf{W}_T$ is uniformly positive definite and hence the linear system $(\mathbf{A};\mathbf{B})$ is exactly controllable.

V. Linear Quadratic Regulation (LQR) of Infinite Dimensional Network Systems

A. The LQR Problem

Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and the inner product in $L^2[0,1]$. For finite $T > 0$, consider the problem of minimizing the cost given by

$$J(\mathbf{u}) = \int_0^T \left[\|\mathbf{C}\mathbf{x}_\tau\|^2 + \|\mathbf{u}_\tau\|^2\right] d\tau + \langle \mathbf{P}_0\mathbf{x}_T, \mathbf{x}_T\rangle \tag{15}$$

over all controls $\mathbf{u} \in L^2(0,T; L^2(0,1))$ subject to the system model constrains in (14). The assumptions for $\mathbf{C}$ and $\mathbf{P}_0$ are:

$$(H2) \quad \begin{cases} \text{(iii)} \quad \mathbf{P}_0 \in \mathcal{L}(L^2[0,1]) \text{ is hermitian and} \\ \qquad \text{non-negative,} \\ \text{(iv)} \quad \mathbf{C} \in \mathcal{L}(L^2[0,1]; Y) \end{cases}$$

where Y is the Hilbert space of observations, which in the current case is $L^2[0,1]$.

Finding the feedback control via dynamic programming consists of the two following steps:

Step 1. Solve the Riccati equation

$$\dot{\mathbf{P}} = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{C}, \quad \mathbf{P}(0) = \mathbf{P}_0 \tag{16}$$

Step 2. Given the solution $\mathbf{P}$ to the Riccati equation, it can be proved that the optimal control $\mathbf{u}^*$ is given by

$$\mathbf{u}_t^* = -\mathbf{B}^T\mathbf{P}(T-t)\mathbf{x}_t^*, \quad t \in [0,T] \tag{17}$$

and moreover that $\mathbf{x}^*$ is the solution of the closed loop equation

$$\begin{aligned} &\dot{\mathbf{x}}_t = \mathbf{A}\mathbf{x}_t - \mathbf{B}\mathbf{B}^T\mathbf{P}(T-t)\mathbf{x}_t, \\ &t \in [0,T], \mathbf{x}_0 \in L^2[0,1]. \end{aligned} \tag{18}$$

B. Existence and Uniqueness of Solutions to LQR Problems

Applying the results in [19] and specializing the Hilbert space there to be $L^2[0,1]$ space, we can show that, under the assumption $(H1)$ and $(H2)$, the existence and uniqueness of the solution to the Riccati equation (16) and the existence and uniqueness of optimal solution pair $(\mathbf{u}^*, \mathbf{x}^*)$ in (17) and (18).

VI. Graphon-Network Regulation of Large-scale Networks

A. Graphon-Network Regulation Strategy

In this work, the basic assumption in the formulation of LQR problems for linear systems distributed on complex networks is that the regulation problem for the infinite dimensional graphon limit systems can be solved (e.g. by established approximation methods) while the finite dimensional LQR problems for the original complex network systems are intractable due to their cardinality.

The proposed control strategy consists of following steps:

1) Consider the control problem of regulating the states of each member of $\{(A_N; B_N)\}_{N=1}^{\infty} \in \mathcal{S}$. Let $\{(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}}) \in \mathbf{G_1^{sp}} \times \mathbf{G_1^{sp}}\}_{N=1}^{\infty}$ be the sequence of step function systems equivalent to $\{(A_N; B_N)\}_{N=1}^{\infty} \in \mathcal{S}$ under the mapping M_G and assume that it converges to the graphon system $(\mathbf{A}; \mathbf{B}) \in \mathbf{G_1^{sp}} \times \mathbf{G_1^{sp}}$ in the L^2 metric.

2) Define the linear quadratic cost for $(\mathbf{A}; \mathbf{B})$ as

$$J(\mathbf{u}) = \int_0^T [\|\mathbf{C}\mathbf{x}_\tau\|^2 + \|\mathbf{u}_\tau\|^2] d\tau + < \mathbf{P}_0 \mathbf{x}_T, \mathbf{x}_T >$$

and define the linear quadratic cost for $(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})$ as

$$J(\mathbf{u^{[N]}}) = \int_0^T [\|\mathbf{C_s^{[N]}} \mathbf{x_t^{[N]}}\|^2 + \|\mathbf{u_t^{[N]}}\|^2] dt + < \mathbf{P_{s0}^{[N]}} \mathbf{x_T^{[N]}}, \mathbf{x_T^{[N]}} >$$

where it is assumed that $\mathbf{C_s^{[N]}} \to \mathbf{C}$ and $\mathbf{P_{s0}^{[N]}} \to \mathbf{P}$ in the strong operator sense. Solve the infinite dimensional Riccati equation for $(\mathbf{A}; \mathbf{B})$ to generate the solution $\mathbf{P}$.

3) Approximate $\mathbf{u}_t = -\mathbf{B}^T \mathbf{P}(T-t)\mathbf{x}_t$ for $(\mathbf{A}; \mathbf{B})$ to generate $\mathbf{u^{[N]}}$ for the finite system $(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})$; Alternatively approximate $\mathbf{P}$ to generate $\tilde{\mathbf{P}}_\mathbf{N}$ and hence the control law $\mathbf{u_t^{[N]}} = -\mathbf{B_s^{[N]}}^T \tilde{\mathbf{P}}_\mathbf{N}(T-t)\mathbf{x_t^{[N]}}$ for $(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})$.

B. Control Law Approximations

There are two ways to generate the control law for finite dimensional systems:

(1) Approximate $\mathbf{u}$ for $(\mathbf{A}; \mathbf{B})$ to generate $\mathbf{u^{[N]}}$ for the finite system $(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})$.

(2) By approximating the Riccati equation solution $\mathbf{P}$ for $(\mathbf{A}; \mathbf{B})$ we can generate $\tilde{\mathbf{P}}_\mathbf{N}$ that provides the control law for finite dimensional network system.

$$\mathbf{u_t^{[N]}} = -\mathbf{B_s^{[N]}}^T \tilde{\mathbf{P}}_\mathbf{N}(T-t)\mathbf{x_t^{[N]}}.$$

C. Small-to-large Approximate Control via Control Input Approximation

Part (1) of control law approximations is proposed and discussed in details in [11]. In the following subsection we add the important results on applying a smaller dimensional control to a larger dimensional system along the convergent sequence of systems $\tilde{S}$.

Theorem 6 ([22]). Consider two system $(\mathbf{A_s^{[N]}}; I)$ and $(\mathbf{A_s^{[M]}}; I)$ $(M > N)$ in a sequence of systems converging to the graphon limit system $(\mathbf{A}; I)$ where I denotes the identity input operator. Denote control law generated via approximate graphon control for $(\mathbf{A_s^{[N]}}; I)$ by $\mathbf{u^{[N]}}$ and that for $(\mathbf{A_s^{[M]}}; I)$ by $\mathbf{u^{[M]}}$. If the initial state for the two systems are of zero L^2 distance, then

$$\begin{aligned} &\|\mathbf{x_T^{[M]}}(\mathbf{u^{[M]}}) - \mathbf{x_T^{[M]}}(\mathbf{u^{[N]}})\|_2 \\ &\qquad \le \|\mathbf{A_s^{[M]}}\|_2 \int_0^T e^{(T-t)} \|\mathbf{u_t^{[M]}} - \mathbf{u_t^{[N]}}\|_2 dt \end{aligned} \tag{19}$$

Furthermore, as $N \to \infty$ and $M \to \infty$,

$$\|\mathbf{x_T^{[M]}}(\mathbf{u^{[M]}}) - \mathbf{x_T^{[M]}}(\mathbf{u^{[N]}})\|_2 \to 0.$$

Theorem 7 ([22]). Consider two system $(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})$ and $(\mathbf{A_s^{[M]}}; \mathbf{B_s^{[M]}})$ $(M > N)$ in a sequence of systems converging to the graphon limit system $(\mathbf{A}; \mathbf{B})$. Denote the control law generated via approximate graphon control for $(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})$ by $\mathbf{u^{[N]}}$ and that for $(\mathbf{A_s^{[M]}}; \mathbf{B_s^{[M]}})$ by $\mathbf{u^{[M]}}$. If the initial state for the two systems are of zero L^2 distance, then

$$\begin{aligned} &\|\mathbf{x_T^{[M]}}(\mathbf{u^{[M]}}) - \mathbf{x_T^{[M]}}(\mathbf{u^{[N]}})\|_2 \\ &\le \|\mathbf{A_s^{[M]}}\|_2 \int_0^T e^{(T-t)} \|\mathbf{B_s^{[M]}}(\mathbf{u_t^{[M]}} - \mathbf{u_t^{[N]}})\|_2 dt \end{aligned} \tag{20}$$

Furthermore, as $N \to \infty$ and $M \to \infty$,

$$\|\mathbf{x_T^{[M]}}(\mathbf{u^{[M]}}) - \mathbf{x_T^{[M]}}(\mathbf{u^{[N]}})\|_2 \to 0.$$

Theorems 6 and 7 imply that the graphon control law generated for a smaller dimensional system can be applied to systems with larger dimensions along a converging sequence of networks. This makes it possible to approximately control extremely large-scale networks via a smaller dimensional control.

D. Approximation of the Solution of the Riccati Equation and Its Convergence Properties

In this subsection, we will discuss part (2) of control law approximations.

1) Basic Notations: Let

$$\Sigma(L^2[0,1])) = \{T \in \mathcal{L}(L^2[0,1])) : T \text{ is hermition}\}$$

and

$$\begin{aligned} &\Sigma^+(L^2[0,1]) \\ &\qquad = \{T \in \Sigma(L^2[0,1]) : (Tx, x) \ge 0, \forall x \in L^2[0,1]\}. \end{aligned}$$

Denote the topological space of all strongly continuous mappings $F : I \to \Sigma(L^2[0,1])$ endowed with strong convergence (see [19]) by $C_s(I; \Sigma(L^2[0,1]))$.

2) Approximation of the Solution of the Riccati Equation: We need to extend the step function approximation to step function approximation with integration against measures.

First, we construct the equivalent representation of the linear operator $\mathbf{P}$ in $C_s([0,T];\Sigma^+(L^2[0,1]))$ by integration against measures. Second, we construct a method to approximate operator $\mathbf{P}$ by integration with respect to measures over partitions. Then we prove that the step function approximation against measures converges in the strong convergence sense. The *step function approximation against measures* of $\mathbf{P}$ is done by integration against measures as follows:

$$\tilde{\mathbf{P}}_{\mathbf{N}}(x,y) = \frac{\int_{S_i \times S_j} \mathbf{P}(x,y) d\sigma(x,y)}{\mu(S_i) \times \mu(S_j)}, \forall (x,y) \in S_i \times S_j, \tag{21}$$

where $S_i, S_j \subset [0,1]$, $\mu(S_i)$ represents the size of the interval S_i and $\sigma(x,y)$ represents the measure (which can be a singular measure, a Lebesgue measure or a mixed measure).

Since $\tilde{\mathbf{P}}_{\mathbf{N}} x$ is the step function approximation of $\mathbf{P}x$ in $L^2[0,1]$ under the interpretation of integration against measures, for any $x \in L^2[0,1]$,

$$\lim_{N\to\infty} \|\tilde{\mathbf{P}}_{\mathbf{N}} x - \mathbf{P}x\|_2 = 0.$$

3) The Approximation of the Riccati Solution and Its Convergence to the Optimal Riccati Solution:

Lemma 2 ([22]). Let $\tilde{\mathbf{P}}_{\mathbf{N}}$ be generated by stepping from $\mathbf{P}$ via $N \times N$ uniform partition of $[0,1]^2$. Then

$$\lim_{N\to\infty} \tilde{\mathbf{P}}_{\mathbf{N}} = \mathbf{P}, \quad \text{in } C_s([0,T];\Sigma(L^2[0,1])).$$

Theorem 8 ([22]). Let $\tilde{\mathbf{P}}_{\mathbf{N}}$ be generated by step function approximation against measures from $\mathbf{P}$ via $N \times N$ uniform partition of $[0,1]^2$. For any $x \in L^2[0,1]$, for any $t \in [0,T]$,

$$\lim_{N\to\infty} \|\tilde{\mathbf{P}}_{\mathbf{N}}(t)x - \mathbf{P}^{[\mathbf{N}]}_{\mathbf{s}}(t)x\|_2 = 0,$$

where $\mathbf{P}^{[\mathbf{N}]}_{\mathbf{s}}$ is the solution of Riccati equation of $(\mathbf{A}^{[\mathbf{N}]}_{\mathbf{s}};\mathbf{B}^{[\mathbf{N}]}_{\mathbf{s}})$ that converges strongly to the solution $\mathbf{P}$.

4) Convergence of the States and Convergence of the Cost: Let $\mathbf{P}^{[\mathbf{N}]}_{\mathbf{s}}$ denote the solution of the Riccati equation for $(\mathbf{A}^{[\mathbf{N}]}_{\mathbf{s}};\mathbf{B}^{[\mathbf{N}]}_{\mathbf{s}})$ that converges strongly to the solution $\mathbf{P}$ of the Riccati equation for $(\mathbf{A};\mathbf{B})$. Let $\tilde{\mathbf{P}}_{\mathbf{N}}$ be the step function approximation against measures for $\mathbf{P}$ generated via the $N \times N$ uniform partition of $[0,1]^2$.

Theorem 9 ([22]). Consider the time horizon $[0,T]$. Let the optimal linear quadratic control law for $(\mathbf{A}^{[\mathbf{N}]}_{\mathbf{s}};\mathbf{B}^{[\mathbf{N}]}_{\mathbf{s}})$ be generated by

$$\mathbf{u}^{N*}_t = -\mathbf{B}^{[\mathbf{N}]^T}_{\mathbf{s}} \mathbf{P}^{[\mathbf{N}]}_{\mathbf{s}}(T-t)\mathbf{x}^{N*}_t,$$

where the optimal state trajectory is given by $\mathbf{x}^{N*}$, and let the graphon approximate control law for $(\mathbf{A}^{[\mathbf{N}]}_{\mathbf{s}};\mathbf{B}^{[\mathbf{N}]}_{\mathbf{s}})$ be

$$\mathbf{u}^{[\mathbf{N}]}_{\mathbf{t}} = -\mathbf{B}^{[\mathbf{N}]^T}_{\mathbf{s}} \tilde{\mathbf{P}}_{\mathbf{N}}(T-t)\mathbf{x}^{[\mathbf{N}]}_{\mathbf{t}},$$

where the corresponding state trajectory is given by $\mathbf{x}^{[\mathbf{N}]}$. Then

$$\forall t \in [0,T], \quad \lim_{N\to\infty} \|\mathbf{x}^{N*}_t - \mathbf{x}^{[\mathbf{N}]}_{\mathbf{t}}\|_2 = 0,$$

and

$$\lim_{N\to\infty} |J(\mathbf{u}^{N*}) - J(\mathbf{u}^{[\mathbf{N}]})| = 0.$$

5) The Small-to-large Approximate Control via Riccati Equation Solution Approximation: We consider the application of a small dimensional regulation law to larger dimensional systems along the convergent sequence. Consider two system $(\mathbf{A}^{[\mathbf{N}]}_{\mathbf{s}};\mathbf{B}^{[\mathbf{N}]}_{\mathbf{s}})$ and $(\mathbf{A}^{[\mathbf{M}]}_{\mathbf{s}};\mathbf{B}^{[\mathbf{M}]}_{\mathbf{s}})$ $(M > N)$ in a sequence of systems converging to the graphon limit system $(\mathbf{A};\mathbf{B})$. Let the respective operators $\mathbf{C}^{[\mathbf{N}]}_{\mathbf{s}}$ and $\mathbf{C}^{[\mathbf{M}]}_{\mathbf{s}}$ in the LQ cost as (15) lie in a sequence of operators converging strongly to $\mathbf{C}$. Similarly, let the respective $\mathbf{P}^{[\mathbf{N}]}_{\mathbf{s0}}$ and $\mathbf{P}^{[\mathbf{M}]}_{\mathbf{s0}}$ in the LQ cost as (15) lie in a sequence of operators converging strongly to $\mathbf{P_0}$. Let $\mathbf{P}^{[\mathbf{M}]}_{\mathbf{s}}$ denote the solution of Riccati equation for $(\mathbf{A}^{[\mathbf{M}]}_{\mathbf{s}};\mathbf{B}^{[\mathbf{M}]}_{\mathbf{s}})$ with LQ cost defined by $\mathbf{C}^{[\mathbf{M}]}_{\mathbf{s}}$ and $\mathbf{P}^{[\mathbf{M}]}_{\mathbf{s0}}$ as in (15). Let $\tilde{\mathbf{P}}_{\mathbf{N}}$ be generated by step function approximation against measures from $\mathbf{P}$ via $N \times N$ uniform partition of $[0,1]^2$.

Theorem 10 ([22]). For any $x \in L^2[0,1]$, for any $t \in [0,T]$,

$$\lim_{N,M\to\infty} \|\tilde{\mathbf{P}}_{\mathbf{N}}(t)x - \mathbf{P}^{[\mathbf{M}]}_{\mathbf{s}}(t)x\|_2 = 0.$$

Theorem 11 ([22]). Consider the time horizon $[0,T]$. Let the optimal linear quadratic control law for $(\mathbf{A}^{[\mathbf{M}]}_{\mathbf{s}};\mathbf{B}^{[\mathbf{M}]}_{\mathbf{s}})$ be generated by

$$\mathbf{u}^{M*}_t = -\mathbf{B}^{[\mathbf{M}]^T}_{\mathbf{s}} \mathbf{P}^{[\mathbf{M}]}_{\mathbf{s}}(T-t)\mathbf{x}^{M*}_t,$$

where the optimal state trajectory is given by $\mathbf{x}^{M*}$, and let the small-to-large control law for $(\mathbf{A}^{[\mathbf{M}]}_{\mathbf{s}};\mathbf{B}^{[\mathbf{M}]}_{\mathbf{s}})$ be generated by

$$\mathbf{u}^{[\mathbf{M},\mathbf{N}]}_{\mathbf{t}} = -\mathbf{B}^{[\mathbf{M}]^T}_{\mathbf{s}} \tilde{\mathbf{P}}_{\mathbf{N}}(T-t)\mathbf{x}^{[\mathbf{M},\mathbf{N}]}_{\mathbf{t}},$$

where the corresponding state trajectory is given by $\mathbf{x}^{[\mathbf{M},\mathbf{N}]}$. Then

$$\forall t \in [0,T], \quad \lim_{N,M\to\infty} \|\mathbf{x}^{M*}_t - \mathbf{x}^{[\mathbf{M},\mathbf{N}]}_{\mathbf{t}}\|_2 = 0,$$

and

$$\lim_{N,M\to\infty} |J(\mathbf{u}^{M*}) - J(\mathbf{u}^{[\mathbf{M},\mathbf{N}]})| = 0.$$

VII. Simulation Examples

Consider a network system evolving according to the node averaging dynamics with weighted graph G_N describing the dynamic interactions. Suppose each node has an independent input channel. Denote the system by $(A_N; I_N)$, where A_N is the adjacency matrix of G_N and I_N is the identity input mapping. The network system $(A_N; I_N)$ with (normalized) node dynamics is therefore described by

$$\dot{x}^i_t = \frac{1}{N}\sum_{j=1}^{N} A_{Nij} x^j_t + u^i_t, \quad x^i_t, u^i_t \in R, i \in \{1,...,N\}. \tag{22}$$

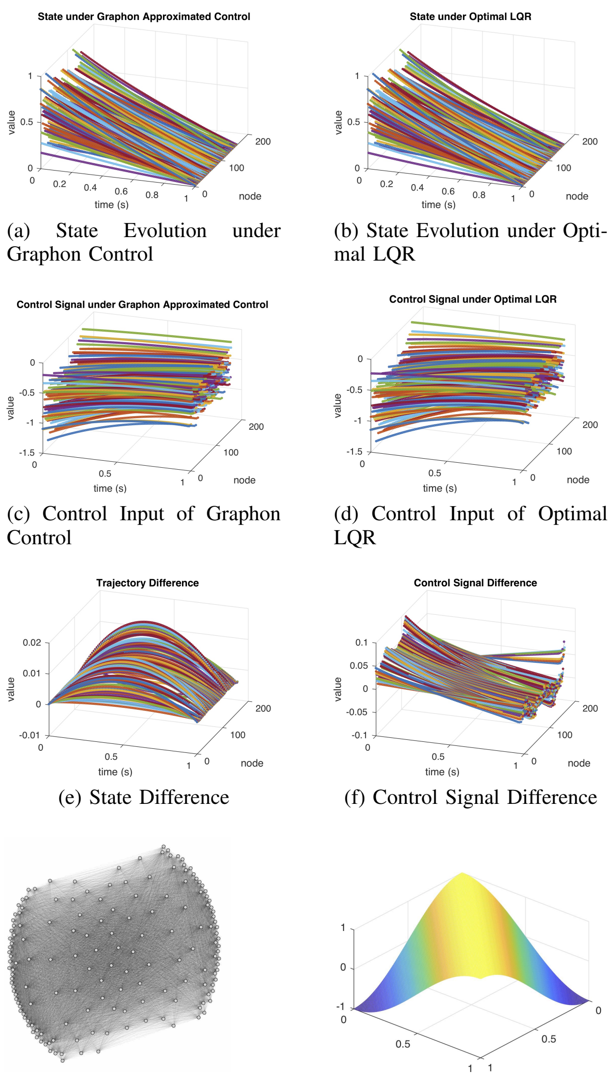
The regulation objective is to regulate the network states around origin from random initial states with minimum linear quadratic cost.

As an example, we consider a sequence of networks converging to the graphon limit $\mathbf{U}(x,y) = \cos(\pi(x-y))$ for all $x, y \in [0,1]$ as in figure (h) and solve the LQR problem over the time horizon $[0, T]$ for the network sequence. (See [11] for detailed description of the generation of a convergent network sequence).

Naturally, in the application of the grapoh-LQR methodology, a finite complex network G_N is not generated via a hidden graphon. A plausible empirical approach to model the required infinite limit graphon G_∞ is to fit two dimensional Fourier series to the step function representation of the adjacency matrix. Such parametric modelling of empirical data could resemble parametric estimation in statistics and system identification. These topics are the subjects of current research [22].

A. Graphon-LQR Example

In this simulation, as shown in Figure 4, a network of size 160 along the sequence is considered. The system is represented by (A_{160}, I_{160}) with A_{160} as the adjacency matrix of the weighted network and I_{160} as the identity input matrix of size 160. The simulation time horizon is $[0, 1]$ and the parameters used in the LQ cost are $Q = C^T C = I_{160}$ and $P_0 = 100 I_{160}$. The control law is generated by approximating the Riccati equation solution as in (21). Both the graphon-LQR control and the LQR optimal control regulate the system from the same random initial states to the origin as shown in figures (a) and (b). From figures (e) and (f), we see that the graphon-LQR control achieves remarkably similar performance to the LQR optimal control. The maximum trajectory difference from the optimal control is less than 2% of the maximum initial states.

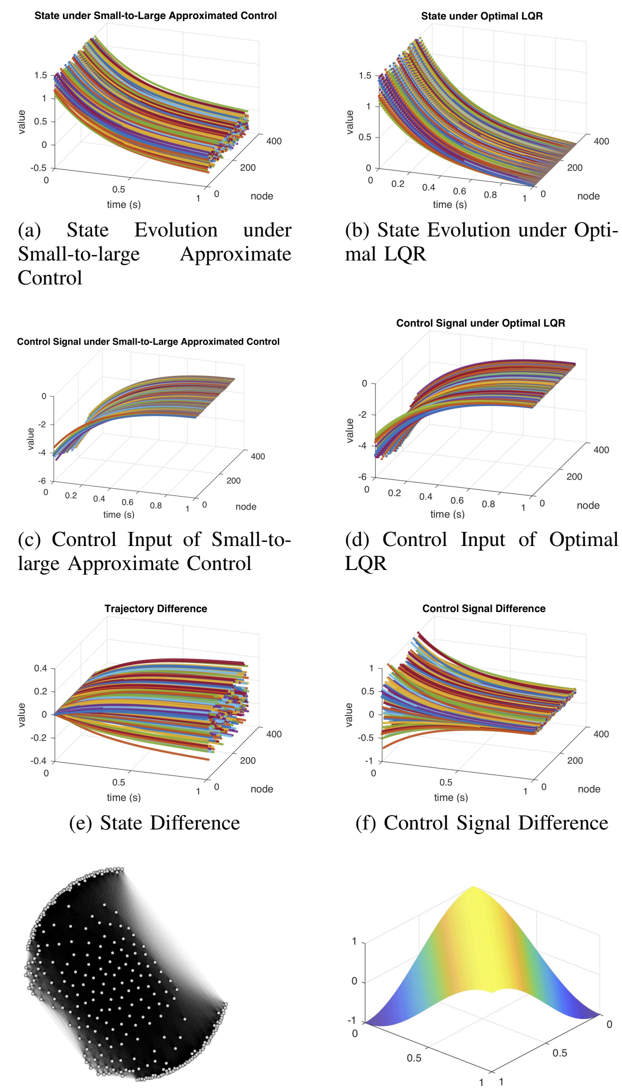


(a) State Evolution under Graphon Control
(b) State Evolution under Optimal LQR
(c) Control Input of Graphon Control
(d) Control Input of Optimal LQR
(e) State Difference
(f) Control Signal Difference
(g) Network of 160 Nodes
(h) Graphon Limit

Fig. 4. Simulation on a Network of 160 Nodes

B. Small-to-large Approximate Regulation Example

In this simulation, we apply the regulation law generated for a smaller dimensional system to a larger dimensional system along the converging sequence. The smaller dimensional network is of size 80 and the larger dimensional network is of size 320. The large dimensional system is represented by (A_{320}, I_{320}) with A_{320} as the adjacency matrix of the



(a) State Evolution under Small-to-large Approximate Control
(b) State Evolution under Optimal LQR
(c) Control Input of Small-to-large Approximate Control
(d) Control Input of Optimal LQR
(e) State Difference
(f) Control Signal Difference
(g) Network of 320 Nodes
(h) Graphon Limit

Fig. 5. Simulation on a Network of 320 Nodes

weighted network and I_{320} as the identity input matrix of size 320. The simulation time horizon is $[0, 1]$ and the parameters used in the LQ cost are $Q = C^T C = 10I_{320}$ and $P_0 = 100I_{320}$. As the result in Figure 5 shows, the small-to-large approximate control approximate well on the large dimensional system. The trajectory difference depends on the initial states of the large dimensional system. In this simulation example, each 4 adjacent nodes (in labelling) receive the same control input signal due to the approximation in the Riccati equation solution. Therefore, differences in the initial conditions of the adjacent 4 nodes give rise to a difference in the terminal states. However the mean effect is subject to the LQR control law.

VIII. Conclusion

We propose a methodology to approximately regulate network systems using graphons. Important aspects which require further investigation include: (1) the application of the regulation strategy to asymmetric (i.e., directed) network systems; (2) an equivalent theory to that in this paper for sparse networks; (3) fitting 2D analytic models (e.g. Fourier series, etc) to empirical data in order to provide parameterized models for approximating limiting graphons. Finally, this paper only deals with centralized control, while the decentralized control of complex systems is formulated within a graphon theoretic mean field games framework in [23].

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