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Low-dimensional solutions for optimal control of network-coupled subsystems over a directed network

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***Abstract*— In this paper, we investigate optimal control of network-coupled subsystems, where the coupling between the dynamics of the subsystems is represented by the adjacency or Laplacian matrix of a directed graph. Under the assumption that the coupling matrix is normal and the cost coupling is compatible with the dynamics coupling, we use the spectral decomposition of the coupling matrix to decompose the overall system into at most n systems with noise coupled dynamics and decoupled cost, where n is the size of the network. Furthermore, the optimal control input at each subsystem can be computed by solving n_1 decoupled Riccati equations where n_1 $(n_1 \le n)$ denotes the number of distinct eigenvalues of the coupling matrix, where complex conjugate pairs are not double-counted. A salient feature of the result is that the solution complexity depends on the number of distinct eigenvalues of the coupling matrix rather than the size of the network. Therefore, the proposed solution framework provides a scalable method for synthesizing and implementing optimal control laws for large-scale network-coupled subsystems.**

I. Introduction

A. Background and Motivation

Many modern technological systems involve many subsystems connected over networks with multiple sensors and actuators. Such network systems appear in smart grids, Internet of Things, and autonomous vehicle fleets, among others. For large network systems, it is important to identify conditions that allow low-complexity control synthesis and implementation. These conditions typically involve simplifications in control structures (e.g., pinning control [1] and ensemble control [2]), control objectives (e.g., consensus [3]–[5] and synchronization [6]), and couplings among subsystems (e.g., symmetric interconnections [7]–[10], exchangeable or anonymous subsystems [11]–[14], sparse connections or structure reduction [15], [16], hierarchical decompositions [17] and patterned systems [18]), as well as approximations in optimality and control (e.g., mean-field games [19], [20], control based on approximate aggregations [21], approximate distributed control [22], [23], and graphon control [24]).

Spectral decompositions for controlling large-scale systems have been used in different problem formulations. An earlier work [21] considered the problem of approximating a high-dimensional system with a low-dimensional system using state aggregation. Spectral decomposition of large-scale systems with symmetric interconnected subsystems have been considered in [10], [25]. Algebraic decomposition of mean-field coupled subsystems has been considered in [12]–[14]. Spectral decomposition of large network-coupled systems with heterogeneous couplings over an undirected network has been investigated in [26]. Similar decomposition for controlling systems coupled over large undirected graphs and graph limits has been investigated in [27].

The goal of the current paper is to extend results in [26] to analyze systems coupled over directed networks. Control of multiple subsystems over directed graphs has also been investigated extensively by many researchers (e.g. in the context of consensus problems [28], [29] and cooperative optimal control [29]–[31]) where the graphs typically represent the underlying communication networks. These are different problems compared to the current paper.

The main idea of [26] is to leverage the spectral decomposition of the coupling matrix to reduce the computational complexity of control. When the coupling corresponds to an undirected graph, the coupling matrix is real and symmetric, and therefore all the eigenvalues are real and non-negative. This feature was exploited in [26] to decompose the original system into decoupled eigen-directions. The same idea does not work in the directed case because the coupling matrix is not necessarily symmetric and hence the eigenvalues are complex, in general. Therefore, naively using the approach of [26] will result in the cost along each eigen-direction to be complex valued, leading to an ill posed optimal control problem.

B. Contributions of this Article

In this paper, we investigate a control system where the coupling between the dynamics of the subsystems is represented by the adjacency or Laplacian matrix of a directed graph. Each subsystem has a local state and takes a local control action. The evolution of the state of each subsystem depends on its local state and local control as well as a weighted combination (which we call the network field) of the states and controls of its neighbors. Those weighted combinations from the network field are encoded in the coupling matrix. This paper focuses on the case where the coupling matrix is normal, in which case it has a spectral decomposition. The objective is to choose the control inputs of each subsystem to minimize the total cost over time. The above model is a linear quadratic regulation problem and a centralized solution can be obtained by solving an $nd_x \times nd_x$-dimensional Riccati equation, where n is the number of subsystems and d_x is the dimension of the state of each

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subsystem. We propose an alternative solution that has low complexity and may be implemented in a local manner with projected state information. The most important difference with the work in [26] is that real normal matrices have a complex spectral decomposition, meaning that the dynamics and cost need to be expressed in terms of a well-posed in a complex LQR problem.

The main contributions of our paper are the following:

- A spectral decomposition technique is developed to decompose the linear quadratic control problem for network-coupled dynamical subsystems into n decoupled subproblems, where n is the number of subsystems.
- These n_0 decoupled subproblems can be solved by solving only n_1 decoupled Riccati equations of dimension $d_x \times d_x$, where n_1 denotes the number of distinct, eigenvalues of the coupling matrix (without double-counting conjugate pairs and duplicate eigenvalues), and d_x is the state dimension of each subsystem. In contrast, a direct centralized solution requires solving an $nd_x \times nd_x$-dimensional Riccati equation where n is the number of subsystems. We note that for any coupling matrix, the inequalities $n_1 \leq n_0 \leq n$ always hold. Thus the method proposed in this paper leads to considerable simplifications in synthesizing optimal control laws.
- The solution method is applied to study networks coupled by a circulant matrix, meaning that the locally-perceived coupling is node-invariant.

C. Notations and Definitions

Let $\mathbb{R}$ denote the set of real numbers. The notation $A = [a_{ij}]$ means that a_{ij} is the $(i,j)^{\text{th}}$ element of the matrix A. For a vector v, v_i denotes its i^{th} element. For a matrix A, $A^{\intercal}$ denotes its transpose, $A^{\dagger}$ denotes its Hermitian, and $\operatorname{Tr}(A)$ denotes its trace. Given vectors $v^1, \ldots, v^n$, $\operatorname{cols}(v^1, \ldots, v^n)$ denotes the matrix formed by horizontally stacking the vectors. For any natural number n, $\mathbb{I}_n$ denotes the n-dimensional identity matrix, $\mathbb{1}_{n\times n}$ denotes the $n \times n$-dimensional matrix of ones, and $\mathbb{1}_n$ denotes the n-dimensional vector of ones. $\mathbb{E}$ denotes the expectation. $x(0{:}T)$ denotes the family $\{x(t),\ t \in \{0,\ldots,T\}\}$. $\sqrt{-1}$ is used for the imaginary unit. We define a sesquilinear form on $\mathbb{C}^{d\times n}$ with weight $P = [p_{ij}] \in \mathbb{C}^{n\times n}$ as follows: for any $x = \operatorname{cols}(x_1, \ldots, x_n), y = \operatorname{cols}(y_1, \ldots, y_n) \in \mathbb{C}^{d\times n}$,

$$\langle x, y\rangle_P = \sum_{i,j\in\mathcal{N}} p_{ij} x_i^{\dagger} y_j = \operatorname{Tr}(x^{\dagger} y P). \tag{1}$$

Two equivalent characterizations of this map are

$$\langle x, y\rangle_P = \sum_{i\in\mathcal{N}} x_i^{\dagger} y P_i \quad \text{and} \quad \langle x, y\rangle_P = \sum_{j\in\mathcal{N}} P_j^{\intercal} x^{\dagger} y_j,$$

where $P_i \in \mathbb{C}^n$ denotes the i^{th} column of P. When P is equal to its Hermitian and positive definite, $\langle\cdot,\cdot\rangle_P$ is an inner product on $\mathbb{C}^{d\times n}$.

For a linear system with (possible complex-valued[1]) system matrices A, B, per-step cost matrices Q, R, and terminal cost matrices Q_T of compatible dimensions, we use the notation $P(0{:}T) = \mathcal{R}_T(A, B, Q, R, Q_T)$ to denote the solution $P(0{:}T))$ of the backward Riccati difference equation initialized at $P(T) = Q_T$ and for $t \in \{0, \ldots, T-1\}$, computed recursively using

$$\begin{aligned} P(t) &= Q + A^{\dagger} P(t{+}1) A \\ &\qquad - A^{\dagger} P(t{+}1) B \big(R + B^{\dagger} P(t{+}1) B\big)^{-1} B^{\dagger} P(t{+}1) A, \end{aligned}$$

and the notation $K(0{:}T{-}1) = \mathcal{K}_T(A, B, Q, R, Q_T)$ to denote the sequence of gains $K(0{:}T{-}1)$ obtained using

$$K(t) = -\big(R + B^{\dagger} P(t+1) B\big)^{-1} B^{\dagger} P(t+1) A.$$

D. Background on Normal Matrices

A square matrix $M \in \mathbb{R}^{n\times n}$ is called normal if $MM^{\intercal} = M^{\intercal}M$. Examples include orthogonal, symmetric, skew-symmetric, and circulant matrices.

A key property of normal matrices is that their eigenvectors can be written to be orthogonal, thus they have a spectral decomposition of the following form

$$M = \sum_{\ell\in\mathcal{N}} \lambda^{\ell} v^{\ell} (v^{\ell})^{\dagger}$$

where $\{\lambda^1, \ldots, \lambda^n\}$ are the eigenvalues of M and $\{v^1, \ldots, v^n\}$ are the corresponding set of *orthonormal* eigenvectors. Since M is real, the eigenvalues are either real or occur in complex conjugate pairs. Moreover, the eigenvector of a real eigenvalue can be chosen to be real while preserving orthogonality and eigenvectors of complex conjugate eigenvalues are complex conjugates of each other.

In our analysis, we are interested in normal matrices that commute. The following lemma states equivalent characterizations of such matrices.

Lemma 1 *Let $M_1, M_2 \in \mathbb{R}^{n\times n}$ be normal. Then, the following statements are equivalent:*

1) *M_1 and M_2 commute (i.e., $M_1 M_2 = M_2 M_1$).*
2) *M_1 and M_2 are simultaneously diagonalizable (i.e., there exists an orthogonal matrix $P \in \mathbb{C}^{n\times n}$ such that $P^{\dagger} M_1 P$ and $P^{\dagger} M_2 P$ are both diagonal matrices).*
3) *M_1 and M_2 share a same set of n orthonormal eigenvectors, where any eigenvalue of M_1 is real if and only if the corresponding eigenvalue of M_2 is real.*

II. System Model and Problem Formulation

In this section, we present the model of network coupled subsystems. The model is similar to the model proposed in [26] with two differences: (i) we consider the coupling of the subsystems to be directed, while the coupling in [26] was undirected. (ii) We also consider the model to be in discrete

[1] In the standard literature on linear quadratic regulation it is assumed that the state, control, the system dynamics, and the cost are real-valued but all the standard results continue to hold for complex-valued states, control, and dynamics as long as the per-step cost is real-valued and lower bounded; provided we replace transposes by conjugate transpose. Such results can be obtained by adapting the proof steps in [32], [33].

time, while the model in [26] was in continuous time, though this does not affect proposed solution.

A. System Model

Consider a network made of n nodes labeled $\mathcal{N} = \{1, \dots, n\}$. Let $M = [m_{ji}] \in \mathbb{R}^{n \times n}$ denote a coupling matrix that corresponds to the connection between the nodes. This connected network may be visualized using a directed graph. For simplicity, the system is said to evolve in discrete time, though the analysis remains unchanged in the continuous-time case. The time horizon will also be taken to be finite, denoting it as $\{0, \dots, T-1\}$.

There is a subsystem associated with each node of the graph. Let $x_i(t) \in \mathbb{R}^{d_x}$, $u_i(t) \in \mathbb{R}^{d_u}$, and $\xi_i(t) \in \mathbb{R}^{d_x}$ denote respectively the state, control input, and noise of node $i \in \mathcal{N}$ at time t. The system starts from a known initial state, $x(0) = \{x_i(0),\ i \in \mathcal{N}\}$, and for any time $t \in \{0, 1, \dots, T-1\}$, the state at node i follows

$$x_i(t+1) = Ax_i(t) + Bu_i(t) + Dx_i^{\mathcal{G}}(t) + Eu_i^{\mathcal{G}}(t) + \xi_i(t) \quad (2)$$

where A, B, D, and E are matrices of appropriate dimensions[2] that do not depend on i, and $\{\xi_i(t)\}_{t\geq 0}$ is an independent and identically distributed noise process with zero mean and finite variance Ξ_i. Moreover, the processes $\{\xi_i(t)\}_{t\geq 0}$, $i \in \mathcal{N}$ are independent.

Define the network field as

$$x_i^{\mathcal{G}}(t) = \sum_{j\in\mathcal{N}} x_j(t) m_{ji} \text{ and } u_i^{\mathcal{G}}(t) = \sum_{j\in\mathcal{N}} u_j(t) m_{ji}, \quad (3)$$

to denote the locally perceived state and the control of the network at node i, The weight m_{ji} quantifies the influence of node j on node i. Following [26], we adopt the atypical notation of describing the system state and system control outputs as a matrix rather than a vector. In particular, define

$$\begin{aligned} x(t) &:= \operatorname{cols}(x_1(t), \dots, x_n(t)) \in \mathbb{R}^{d_x \times n}, \\ x^{\mathcal{G}}(t) &:= \operatorname{cols}(x_1^{\mathcal{G}}(t), \dots, x_n^{\mathcal{G}}(t)) \in \mathbb{R}^{d_x \times n}, \\ u(t) &:= \operatorname{cols}(u_1(t), \dots, u_n(t)) \in \mathbb{R}^{d_u \times n}, \\ u^{\mathcal{G}}(t) &:= \operatorname{cols}(u_1^{\mathcal{G}}(t), \dots, u_n^{\mathcal{G}}(t)) \in \mathbb{R}^{d_u \times n}, \\ \xi(t) &:= \operatorname{cols}(\xi_1(t), \dots, \xi_n(t)) \in \mathbb{R}^{d_x \times n}. \end{aligned}$$

We may write $x^{\mathcal{G}}(t) = x(t)M$, $u^{\mathcal{G}}(t) = u(t)M$. Given an initial state $x(0)$ the dynamics for the system state can be written for $t \geq 0$ as:

$$x(t+1) = Ax(t) + Bu(t) + Dx^{\mathcal{G}}(t) + Eu^{\mathcal{G}}(t) + \xi(t). \quad (4)$$

B. System Performance and Control Objective

At time $t \in \{0, 1, \dots, T-1\}$, the system incurs an instantaneous cost

$$\begin{aligned} c(x(t), u(t)) &= \sum_{i,j\in\mathcal{N}} \left[q_{ij} x_i(t)^{\intercal} Q x_j(t) + r_{ij} u_i(t)^{\intercal} R u_j(t) \right] \\ &= \langle x(t), Qx(t) \rangle_{M_q} + \langle u(t), Ru(t) \rangle_{M_r}. \end{aligned} \quad (5)$$

[2]In principle, the system matrices can vary with time, but we restrict ourselves to the time-invariant setting for simpler notation.

and at the terminal time T , the system incurs a terminal cost

$$c_T(x(T)) = \sum_{i,j\in\mathcal{N}} q_{ij} x_i(T)^{\intercal} Q_T x_j(T) = \langle x(T), Q_T x(T) \rangle_{M_q} \quad (6)$$

where Q, Q_T, and R are matrices of appropriate dimensions and $M_q = [q_{ij}]$ and $M_r = [r_{ij}]$ belong to $\mathbb{R}^{n \times n}$.

We assume that there is a controller that observes the system state $x(t)$ and chooses the control action $u(t)$ according to a state feedback control law $g = (g_0, \dots, g_{T-1})$, i.e, for $t \in \{0, \dots, T-1\}$, we have

$$u(t) = g_t(x(t)). \quad (7)$$

The performance of a control law g is quantified by the expected total cost which is given by

$$J(g) = \mathbb{E}\left[\sum_{t=0}^{T-1} c(x(t), u(t)) + c_T(x(T)) \right]. \quad (8)$$

We are interested in the following optimization problem.

Problem 1 *Choose a control policy $g = (g_1, \dots, g_{T-1})$ of the form* (7) *to minimize the cost $J(g)$ given by* (8)*, subject to the dynamics described in* (2)*.*

This problem can be solved centrally using standard LQR theory, which requires solving a $nd_x \times nd_x$-dimensional Riccati equation, whose naive solution is $\mathcal{O}(n^3 d_x^3)$. Our main result is to present a scalable solution which requires solving at most n distinct $d_x \times d_x$-dimensional Riccati equations, which has a worst case complexity of $\mathcal{O}(n d_x^3)$.

III. Spectral Decomposition of the System

A. Assumptions on the Coupled Dynamics

We impose the following assumptions on the model.

(A0) The coupling matrix $M \in \mathbb{R}^{n \times n}$ is normal.

In the setting of [26], the network was assumed to be undirected, in which case the matrix M is symmetric and assumption (A0) automatically holds. However, Assumption (A0) also holds in more general settings. An important class of normal matrices are *circulant matrices*, which may be viewed as the adjacency matrix of a circulant graph. See [34] for a discussion and Sec. V for an example.

(A1) The weight matrices M_q, M_r both commute with M.

One instance where (A1) is true is when M_q and M_r are polynomials of M (with real-valued coefficients). The intuition behind taking powers of M in a directed, weighted graph is that $(M^k)_{ij}$ represents the total weight of all paths of length k from node i to node j. Each entry accounts for the contributions of different paths of length k in the graph.

We now present some implications of these assumptions. Assumption (A1) with Lemma 1 implies that M, M_q, and M_r share a common set of orthonormal eigenvectors. Since M, M_q, and M_r admit a spectral decomposition, we have

$$M = \sum_{\ell\in\mathcal{N}} \lambda^{\ell} v^{\ell} (v^{\ell})^{\dagger}, \quad (9a)$$

$$M_q = \sum_{\ell\in\mathcal{N}} q^{\ell} v^{\ell} (v^{\ell})^{\dagger}, \qquad M_r = \sum_{\ell\in\mathcal{N}} r^{\ell} v^{\ell} (v^{\ell})^{\dagger} \quad (9b)$$

B. Spectral Decomposition of the Dynamics

We now show that the spectral decomposition of M provided in (9) can be leveraged to obtain a spectral decomposition of the dynamics. For that matter, define the *eigenstates* $x^\ell(t)$, *eigencontrols* $y^\ell(t)$, and *eigennoise* $\xi^\ell(t)$, $\ell \in \mathcal{N}$, as follows

$$x^\ell(t) = x(t)v^\ell(v^\ell)^\dagger, \qquad u^\ell(t) = u(t)v^\ell(v^\ell)^\dagger, \tag{10}$$

$$\xi^\ell(t) = \xi(t)v^\ell(v^\ell)^\dagger. \tag{11}$$

The eigenstates and eigencontrols capture the coupling between the subsystems in the following sense.

Lemma 2 (Network field decomposition) *The following relationships hold:*

$$x^\mathcal{G}(t) = \sum_{\ell \in \mathcal{N}} \lambda^\ell x^\ell(t) \text{ and } u^\mathcal{G}(t) = \sum_{\ell \in \mathcal{N}} \lambda^\ell u^\ell(t). \tag{12}$$

PROOF We present the proof for x, the proof for u is identical. The spectral decomposition of M in (9) gives:

$$x^\mathcal{G}(t) = x(t)M = \sum_{\ell \in \mathcal{N}} \lambda^\ell x(t)v^\ell(v^\ell)^\dagger = \sum_{\ell \in \mathcal{N}} \lambda^\ell x^\ell(t). \tag{13}$$

∎

A key implication of Lemma 2 is the following. Let $x^\ell(t) = \text{cols}(x^\ell_i(t), \dots, x^\ell_i(t))$ and $u^\ell(t) = \text{cols}(u^\ell_i(t), \dots, u^\ell_i(t))$.

Proposition 1 *The local state and control at each node $i \in \mathcal{N}$ may be decomposed as*

$$x_i(t) = \sum_{\ell \in \mathcal{N}} x^\ell_i(t), \;\; u_i(t) = \sum_{\ell \in \mathcal{N}} u^\ell_i(t), \tag{14}$$

where the dynamics of $x^\ell_i(t)$ depends on only $u^\ell_i(t)$ and are

$$x^\ell_i(t+1) = (A + \lambda^\ell D)x^\ell_i(t) + (B + \lambda^\ell E)u^\ell_i(t) + \xi^\ell_i(t) \tag{15}$$

PROOF The relationships (14) follow from definition. For the dynamics (15), observe that Lemma 2 implies that

$$x^\mathcal{G}(t)v^\ell v^{\ell\dagger} = x(t)Mv^\ell(v^\ell)^\dagger = \lambda^\ell x(t)v^\ell(v^\ell)^\dagger, \qquad ∎$$

with a similar relationship holding for $u^\mathcal{G}(t)$.

A similar eigendecomposition was presented in [26], where it was assumed that M is symmetric and, therefore, the eigenvectors $\{v^1, \dots, v^n\}$ were real. In our setting, M is not symmetric, so the eigenvectors $\{v^1, \dots, v^n\}$ are complex, in general. Hence, the eigenstates $\{x^1(t), \dots, x^n(t)\}$ and eigencontrols $\{u^1(t), \dots, u^n(t)\}$ are also complex, in general. Our main result is to present a framework to handle such complex-valued eigenstates and eigencontrols.

C. Assumptions on the Coupled Cost

Two assumptions must be made to have a well-defined cost function:

(A2) For $\ell \in \mathcal{N}$, $\text{Re}(q^\ell) \geq 0$ and $\text{Re}(r^\ell) > 0$, where q^ℓ and q^ℓ are respectively the eigenvalues of M_q and M_r associated with eigenvector v^ℓ.

(A3) The matrices Q and Q_T are symmetric and positive semidefinite, and R is symmetric and positive definite.

We now present the implication of these assumptions. As already mentioned, (A1) implies that $\{v^1, \dots, v^n\}$ are eigenvectors of M_q and M_r. For $\ell \in \mathcal{N}$, the eigenvalues of M_q and M_r corresponding to v^ℓ are q^ℓ and r^ℓ, respectively. By (A2), $\frac{1}{2}(M_q + M_q^\intercal)$ and $\frac{1}{2}(M_r + M_r^\intercal)$ are respectively symmetric positive semidefinite and symmetric positive definite. Hence, for any $y \in \mathbb{C}^{d \times n}$, $\langle y, y \rangle_{(M_q + M_q^\intercal)} \geq 0$, and $\langle y, y \rangle_{(M_r + M_r^\intercal)} > 0$.

Additionally, (A3) ensures that $(\frac{1}{2}(M_q + M_q^\intercal)) \otimes Q$ and $(\frac{1}{2}(M_q + M_q^\intercal)) \otimes Q_T$ are symmetric positive semidefinite, and $(\frac{1}{2}(M_r + M_r^\intercal)) \otimes R$ is symmetric positive definite. Thus, Problem 1 satisfies the standard assumptions on the per-step cost for the finite-horizon LQR problem to have a unique optimal solution.

D. Spectral Decomposition of the Cost

Since M is real, all its complex eigenvalues occur in complex conjugate pairs. Moreover, eigenvectors corresponding to complex conjugate eigenvalues are complex conjugates of each other. Therefore, we can partition the indices of eigenvalues into three sets: the index set $\mathcal{N}_r$ for real eigenvalues given by

$$\mathcal{N}_r = \{\ell \in \mathcal{N} : \text{Im}(\lambda^\ell) = 0\},$$

the index set $\mathcal{N}_c^+$ for complex eigenvalues with positive imaginary parts given by

$$\mathcal{N}_c^+ = \{\ell \in \mathcal{N} : \text{Im}(\lambda^\ell) > 0\},$$

and the index set $\mathcal{N}_c^-$ for complex eigenvalues with negative imaginary parts given by

$$\mathcal{N}_c^- = \{\ell \in \mathcal{N} : \text{Im}(\lambda^\ell) < 0\},$$

where $\mathcal{N} = \mathcal{N}_r \cup \mathcal{N}_c^+ \cup \mathcal{N}_c^-$. Moreover, for every $\ell^+ \in \mathcal{N}_c^+$, there exists an $\ell^- \in \mathcal{N}_c^-$ such that $(\lambda^{\ell^+}, \lambda^{\ell^-})$ and (v^{ℓ^+}, v^{ℓ^-}) are complex conjugate pairs.

Proposition 2 *The instantaneous cost given by* (5) *can be simplified as follows:*

$$c(x(t), u(t)) = \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{N}} c^\ell(x^\ell_i(t), u^\ell_i(t))$$

where

$$c^\ell(x^\ell_i(t), u^\ell_i(t)) = q^\ell x^\ell_i(t)^\dagger Q x^\ell_i(t) + r^\ell u^\ell_i(t)^\dagger R u^\ell_i(t).$$

A similar simplification holds for the terminal cost (6).

See Section IV for the proof.

The decomposition of Prop. 2 is similar to the cost decomposition obtained in [26]. In [26], the coupling matrix M was symmetric, so all the eigenvalues, and hence the coefficients q^ℓ and r^ℓ were real-valued. The main idea of [26] was then to consider the certainty equivalent version of Problem 1 (i.e., consider the problem where the noise $\xi(t) \equiv 0$), which led to n components with decoupled dynamics and cost. The same approach does not work in our setting because the coupling matrix M is not symmetric and therefore the coefficients q^ℓ and r^ℓ are not real-valued. Thus, the cost objective of component ℓ may be complex-valued, so the cost minimization problem is not well-posed.

Our key result is to leverage the conjugate symmetry of eigenstates and eigencontrol for $\ell \in \mathcal{N}_c^+$ and $\mathcal{N}_c^-$. In particular, as was already stated, for each $\ell^+ \in \mathcal{N}_c^+$, there exists a corresponding $\ell^- \in \mathcal{N}_c^-$ such that the eigenvalues $(\lambda^{\ell^+}, \lambda^{\ell^-})$ and eigenvectors (v^{ℓ^+}, v^{ℓ^-}) form complex conjugate pairs. As a result, the states $x^\ell(t) = x(t)v^\ell(v^\ell)^\dagger$ and control inputs $u^\ell(t) = u(t)v^\ell(v^\ell)^\dagger$ also form complex conjugate pairs for $\ell \in \{\ell^+, \ell^-\}$. By leveraging this conjugate symmetry, we can reduce the cost to real-valued terms by grouping the conjugate pairs together.

Proposition 3 *The component $C_t^\ell := c^\ell(x_i^\ell(t), u_i^\ell(t))$ of the cost identified in Prop. 2 satisfies the following:*

1) *For $\ell \in \mathcal{N}_r$, C_t^ℓ is real.*
2) *Let $\ell^+ \in \mathcal{N}_c^+$ and $\ell^- \in \mathcal{N}_c^-$ be such that $(\lambda^{\ell^+}, \lambda^{\ell^-})$ form a complex conjugate pair. Then, $C_t^{\ell^+}$ and $C_t^{\ell^-}$ are also complex conjugate pairs.*

Consequently, the per-step cost can be written as

$$c(x(t), u(t)) = \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{N}} \bar{c}^\ell(x_i^\ell(t), u_i^\ell(t))$$

where

$$\bar{c}^\ell(x_i^\ell(t), u_i^\ell(t)) = \mathrm{Re}(q^\ell) x_i^\ell(t)^\dagger Q x_i^\ell(t) + \mathrm{Re}(r^\ell) u_i^\ell(t)^\dagger R u_i^\ell(t).$$

PROOF (SKETCH) The proof follows from the conjugate symmetry argument given above. Details are omitted due to space limitations.

E. Main Results

The main result is the following.

Theorem 1 *Let $P^\ell \in \mathbb{C}^{d_x \times d_x}$, $\ell \in \mathcal{N}$, be the solution of the following Riccati equations:*

$$P^\ell(0{:}T) = \mathcal{R}_T(A + \lambda^\ell D, B + \lambda^\ell E, \mathrm{Re}(q^\ell)Q, \mathrm{Re}(r^\ell)R, \mathrm{Re}(q^\ell)Q_T), \quad (16)$$

and $K^\ell(0{:}T) \in \mathbb{C}^{d_u \times d_u \times T}, \ell \in \mathcal{N}$, be the respective gains:

$$K^\ell(0{:}T{-}1) = \mathcal{K}_T(A + \lambda^\ell D, B + \lambda^\ell E, \mathrm{Re}(q^\ell)Q, \mathrm{Re}(r^\ell)R, \mathrm{Re}(q^\ell)Q_T).$$

Then, under assumptions (A1)*,* (A2) *and* (A3)*, the optimal control law for Problem 1 is given, for all $i \in \mathcal{N}$, by*

$$u_i(t) = \sum_{\ell\in\mathcal{N}_r} K^\ell(t) x_i^\ell(t) + 2 \sum_{\ell\in\mathcal{N}_c^+} \mathrm{Re}(K^\ell(t) x_i^\ell(t)). \quad (17)$$

Remark 1 Although the eigenstates $\{x_i^\ell(t)\}_{\ell\in\mathcal{N}}$ depend on the eigenvectors $(v^1, \dots, v^n)$, the Riccati equations (16) depend only on the eigenvalues $(\lambda^1, \dots, \lambda^n)$. Consequently, if the coupling matrix has repeated eigenvalues, e.g. due to the presence of certain symmetries in the graph $\mathcal{G}$, the eigendirections associated with the same eigenvalue satisfy the same Riccati equation. Thus, it suffices to solve only $n_0 := |\mathcal{N}_r| + 2|\mathcal{N}_c^+|$ Riccati equations. This number can be further reduced by noting that the cost terms for conjugate eigenvalues are identical, as shown in the proof of the main result. As a result, the required number of Riccati equations reduces to

$$n_1 := |\mathrm{uniq}\{\lambda^\ell : \ell \in \mathcal{N}_r\}| + |\mathrm{uniq}\{\lambda^\ell : \ell \in \mathcal{N}_c\}|.$$

PROOF (THEOREM 1) Consider the following collections of dynamical systems:

- Eigensystem (ℓ, i), $\ell, i \in \mathcal{N}$, with state $x_i^\ell(t)$, control input $u_i^\ell(t)$, dynamics
$$x_i^\ell(t{+}1) = (A{+}\lambda^\ell D)x_i^\ell(t) + (B{+}\lambda^\ell E)u_i^\ell(t) + \xi_i^\ell(t),$$
and the performance for control policy $g_i^\ell = (g_{i,0}^\ell, \dots, g_{i,T-1}^\ell)$, where $u_i^\ell(t) = g_{i,t}^\ell(x_i^\ell(t))$, given by
$$J_i^\ell(g_i^\ell) = \mathbb{E}\left[\sum_{t=0}^{T-1} \bar{c}^\ell(x_i^\ell(t), u_i^\ell(t)) + \bar{c}_T^\ell(x_i^\ell(T)))\right].$$

Propositions 2 and 3 imply that

$$J(g) = \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{N}} J_i^\ell(g_i^\ell). \quad (18)$$

Note that these subsystems have decoupled cost and dynamics that are simply coupled by the noise. From the certainty equivalence principle, we know that the optimal control law for a stochastic system with noise is the same as that of a deterministic system without noise. Such a deterministic system will have decoupled dynamics. Thus, instead of solving:

(CP) choose a policy g that minimizes $J(g)$,

we can equivalently solve the following optimal control problems:

(CP-ℓ) choose a policy g_i^ℓ that minimizes $J_i^\ell(g_i^\ell)$ for $i \in \mathcal{N}$, $\ell \in \mathcal{N}$

Therefore, given the solutions of Problems **(CP-ℓ)**, we can use the Proposition 1 and choose $u_i(t)$ via (14).

Problem **(CP-ℓ)** for $\ell \in \mathcal{N}_r$ are standard optimal control problems and their solution are given as follows. Let $P^\ell \colon \{0, \dots, T\} \to \mathbb{C}^{d_x \times d_x}$ be as given by (16). Then, for all $i \in \mathcal{N}$, the optimal solution of **(CP-ℓ)** is given by $u_i^\ell(t) = K^\ell(t) x_i^\ell(t)$, $\ell \in \mathcal{N}$,

When $\ell \in \mathcal{N}_c^+ \cup \mathcal{N}_c^-$, we note that the state $x^\ell(t)$ and the control $u^\ell(t)$ are both complex-valued. Nonetheless, **(CP-ℓ)** is a well-posed optimal control problem because the per-step cost $\bar{c}^\ell(x_i^\ell(t), u_i^\ell(t))$ and the terminal cost $\bar{c}_T^\ell(x_i^\ell(T))$ are real-valued and non-negative due to (A2) and (A3). Thus, as argued in footnote 1 on page 2, the optimal control problem **(CP-ℓ)** is well-posed and the standard Riccati equation based optimal control law is optimal. Hence, the optimal solution **(CP-ℓ)** is given by $u_i^\ell(t) = K^\ell(t) x_i^\ell(t)$, for all $\ell \in \mathcal{N}_r$.

Therefore, from (14) we get that the optimal control action is given by

$$u_i(t) = \sum_{\ell\in\mathcal{N}} K^\ell(t) x_i^\ell(t). \quad (19)$$

Now, consider $\ell^+ \in \mathcal{N}_c^+$ and $\ell^- \in \mathcal{N}_c^-$ such that $(\lambda^{\ell^+}, \lambda^{\ell^-})$ form a complex-conjugate pair. We can show via backward induction that the corresponding Riccati gains

$(P^{\ell^+}(t), P^{\ell^-}(t))$ also form a complex conjugate pair and so do the corresponding gains $(K^{\ell^+}(t), K^{\ell^-}(t))$. Then, we can show via forward induction that the eigenstates $(x^{\ell^+}(t), x^{\ell^-}(t))$ and the eigencontrols $(x^{\ell^+}(t), x^{\ell^-}(t))$ also form a complex conjugate pair. Thus,

$$K^{\ell^+}(t)x_i^{\ell^+}(t) + K^{\ell^-}(t)x_i^{\ell^-}(t) = 2\operatorname{Re}(K^{\ell^+}(t)x_i^{\ell^+}(t))$$

Substituting the above in (19) gives (17). ∎

Remark 2 The Riccati equations (16) are significantly simpler to solve than the naive centralized Riccati equation. Each Riccati equation in (16) is of dimension $d_x \times d_x$, while the centralized Riccati equation is of dimension $nd_x \times nd_x$. So, even if the coupling matrix's eigenvalues are distinct (i.e., $n_0 = n$), solving the n Riccati equations (16) of dimension $d_x \times d_x$ is significantly simpler than solving one centralized "n-dimensional" Riccati equation. For graphs where $n_1 \ll n$, these savings become even more drastic.

IV. Proof of Proposition 2

A. Preliminary properties of the state decomposition

Lemma 3 *Let k be a positive integer and $\ell, \ell' \in \mathcal{N}$, then we have the following:*

(P1) $x^\ell(t)M = \lambda^\ell x^\ell(t)$ *and* $u^\ell(t)M = \lambda^\ell u^\ell(t)$.

(P2) $x^\ell(t)M^k = (\lambda^\ell)^k x^\ell(t)$ *and* $u^\ell(t)M^k = (\lambda^\ell)^k u^\ell(t)$.

(P3) $x^\ell(t)M_q = q^\ell x^\ell(t)$ *and* $u^\ell(t)M_r = r^\ell u^\ell(t)$.

(P4) $x(t)M_q = \sum_{\ell\in\mathcal{N}} q^\ell x^\ell(t)$ *and* $u(t)M_r = \sum_{\ell\in\mathcal{N}} r^\ell u^\ell(t)$.

(P5) $\sum_{i\in\mathcal{N}} x_i^\ell(t)^\dagger Q x_i^{\ell'}(t) = \delta_{\ell\ell'} \sum_{i\in\mathcal{N}} x_i^\ell(t)^\dagger Q x_i^{\ell'}(t)$, *where $\delta_{\ell\ell'}$ is the Kronecker delta function.*

(P6) $\sum_{i\in\mathcal{N}} x_i(t)^\intercal Q x_i^\ell(t) = \sum_{i\in\mathcal{N}} x_i^\ell(t)^\dagger Q x_i^\ell(t)$ *and* $\sum_{i\in\mathcal{N}} u_i(t)^\intercal R u_i^\ell(t) = \sum_{i\in\mathcal{N}} u_i^\ell(t)^\dagger R u_i^\ell(t)$

Proof We show the result for $x(t)$. The result for $u(t)$ follows from a similar argument.

Since $v^1, \dots, v^n$ are orthonormal, from (9) we have $v^\ell(v^\ell)^\dagger M = \lambda^\ell v^\ell(v^\ell)^\dagger$, which implies (P1). The proof for (P3) is identical to that of (P1), and (P2) follows from (P1).

(P4) follows from (9) and (P3). To prove (P5), we observe that (10) implies that

$$\sum_{i\in\mathcal{N}} x_i^\ell(t)^\dagger Q x_i^{\ell'}(t) = \sum_{i\in\mathcal{N}} v_i^\ell (v^\ell)^\dagger x(t)^\dagger Q x(t) v^{\ell'} v_i^{\ell'\dagger}$$
$$= \Big(\sum_{i\in\mathcal{N}} v_i^\ell (v_i^{\ell'})^*\Big)(v^\ell)^\dagger x(t)^\dagger Q x(t) v^{\ell'}. \quad (20)$$

Since $v^1, \dots, v^n$ is orthonormal, we get $\sum_{i\in\mathcal{N}} v_i^\ell (v_i^{\ell'})^* = (v^{\ell'})^\dagger v^\ell = \delta_{\ell\ell'}$. Substituting this in (20) completes the proof of (P5). To prove (P6) observe that

$$\sum_{i\in\mathcal{N}} x_i(t)^\intercal Q x_i^\ell(t) = \sum_{i\in\mathcal{N}} x_i(t)^\intercal Q x(t) v^\ell v_i^\ell$$
$$= \sum_{i\in\mathcal{N}} v_i^\ell x_i(t)^\intercal Q x(t) v^\ell = (v^\ell)^\dagger x(t)^\intercal Q x(t) v^\ell. \quad (21)$$

From (20), we get that the expression in (21) is equal to $\sum_{i\in\mathcal{N}} x_i^\ell(t)^\dagger Q x_i^\ell(t)$. ∎

B. Proof for Proposition 2

From (1) and (P3), we obtain

$$\Big\langle \sum_{\ell\in\mathcal{N}} x^\ell(t), Q\Big(\sum_{\ell\in\mathcal{N}} x^\ell(t)\Big)\Big\rangle_{M_q}$$
$$= \sum_{i\in\mathcal{N}}\sum_{\ell\in\mathcal{N}} x_i^\ell(t)^\dagger Q\Big(\sum_{\ell'=1}^{n} q^{\ell'} x_i^{\ell'}(t)\Big)$$
$$= \sum_{\ell\in\mathcal{N}}\sum_{i\in\mathcal{N}} x_i^\ell(t)^\dagger Q\Big(\sum_{\ell'=1}^{n} q^{\ell'} x_i^{\ell'}(t)\Big)$$
$$\overset{(b)}{=} \sum_{\ell\in\mathcal{N}}\sum_{i\in\mathcal{N}} q^\ell x_i^\ell(t)^\dagger Q x_i^\ell(t), \quad (22)$$

where (b) follows from (P5). The same holds with cost incurred by the control input u, proving the claim.

V. An Illustrative Example

Consider a n-node network connected over a directed cycle graph $\mathcal{G}$ shown in Fig. 1. Its adjacent matrix is the sparse $n \times n$ matrix with 1's above the diagonal and in the bottom left, and 0 elsewhere. M is a circulant matrix with eigenvalues $\lambda^\ell = e^{-2\pi\sqrt{-1}(\ell-1)/n}$, $\ell \in \mathcal{N}$, being the n roots of unity.

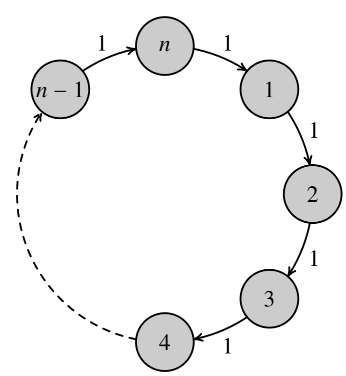

Fig. 1. Directed n-node circle graph.

Consider a circle graph with $n = 9$ for which case we have $M = [m_{ij}]$ with $m_{ij} = 1$ if $j \equiv i + 1 \pmod n$ and 0 otherwise. We assume each subsystem is scalar, thus $d_x = d_u = 1$, and assume that the system matrices are $A = 1$, $B = 2$, $D = 1$, and $E = 2$, with weighting matrices $Q = 5$, $R = 1$, and $Q_T = 10$; Gaussian noise with time-invariant distribution $\xi_i(t) \sim \mathcal{N}(0, 0.02\mathbb{I}_9)$, a random initial condition with distribution $\mathcal{N}(0, \mathbb{I}_9)$, time horizon $T = 40$; and the weight matrices $M_q = 5I_3 + 2M + 3M^2$ and $M_r = I_3$.

In this case, $|\mathcal{N}_r| = 1$ and $|\mathcal{N}_c^+| = 4$. So, we need to solve $n_1 = 5$ separate 1×1 Riccati equations need to be solved (instead of a 9×9 Riccati equation needed for the centralized solution). The computational savings are more drastic for larger n. Two of the resulting trajectories of the global state, and eigenstates, and the corresponding control inputs are shown in Fig. 2.

VI. Conclusion

In this paper, we investigate the optimal control of network-coupled subsystems where the dynamics and cost are coupled via a weighted coupling matrix corresponding to a directed graph. Under the assumption that the coupling matrix is normal, we provide a low-dimensional decomposition of the optimal control problem by projecting the state $x(t)$ into n orthogonal eigendirections, leading to components that are decoupled in the cost and only coupled via the noise in the dynamics, enabling the computation of optimal control

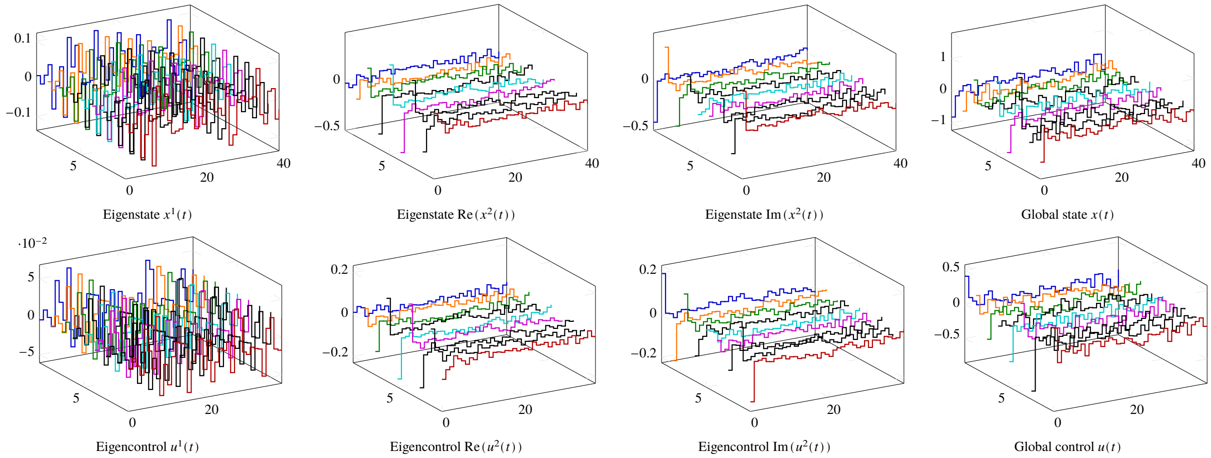

Fig. 2. Plots of eigencomponents $x^\ell(t), u^\ell(t)$, $\ell \in \{1, 2\}$, and global components $x(t)$, $u(t)$.

inputs for each component by solving the n decoupled Riccati equations, leading to considerable computational savings in computing the optimal controller.

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