GMFG Critical Nodes for Control Affine Systems with Exponentiated Costs

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Abstract

Graphon Mean Field Games (GMFGs) [4] constitute generalizations of Mean Field Games for which the agents form subpopulations associated with the nodes of large graphs. The work in ([10], [11]) analyzed the stationarity of equilibrium Nash values with respect to node location for large populations of non-cooperative agents with linear dynamics on large graphs together with their limit graphons. That analysis is extended in this investigation to agent systems lying in the class of control affine non-linear systems (see [15]). Specifically, control affine GMFG systems are treated where (i) at each node $\alpha \in V$ the drift of each generic agent system is affine in the control function, and (ii) the running costs at each node $\alpha \in V \subset \mathbb{R}^m$ are exponentiated negative inverse quadratic (ENIQ) functions of the difference between a generic state and the local graphon weighted mean Z^{α,μ_G}, where $\mu_G := \{\mu_\beta, \beta \in V \subset \mathbb{R}^m\}$ is the globally distributed family of mean fields. Infinite cardinality node and edge limits are considered where it is assumed that the limit graphon $g(\alpha,\beta)$, $(\alpha,\beta) \in V \times V$, is continuous. It is shown that the Nash equilibrium value V^α is stationary with respect to the node location $\alpha \in V$ if and only if the corresponding mean Z^{α,μ_G} is stationary with respect to node location.

Keywords: Mean field games, networks, graphons

1. Introduction

Mean Field Games on graphons is a developing area within the Mean Field Control and Games domain, see for example [3], [4], [17], [9], [21], [7], [22]. The models used in this work generalize those used in standard Mean Field Game theory (see e.g. [5, 6]), where the agents are essentially coupled on complete graphs with uniform weights. This paper employs the Graphon Mean Field Game theory framework introduced in [3], [4] and is focused on the existence and the properties of critical nodes, that is to say nodes at which the solution to the GMFG equations give value functions which are stationary with respect to the graphon parameter. Such nodes constitute stationary Nash value nodes over the infinite limit graph of node locations for games involving large populations of agents distributed over large networks. Initially, [10, 11, 2] analyzed the stationarity of equilibrium Nash values with respect to node location for large populations of non-cooperative agents controlling linear quadratic Gaussian (LQG) systems on large graphs together with their limits, termed graphons. As a follow-up, [12] studied the link between the optimality of nodes and their degrees in the network where degree is interpreted in a suitable limit sense. The initial analysis is extended in this investigation to agent systems lying in the class of control affine non-linear systems (see [15]) with what are termed exponentiated (negative inverse quadratic) (ENIQ) functions.

Consider models of large population games, for which the N agents $\mathcal{A}_i$, $1 \le i \le N < \infty$, are distributed over the finite network, represented by the graph G_k defined by its adjacency matrix $(g^k_{i,j})_{i,j=1:M_k}$. We assume that, at each node of this graph, there is a cluster of agents and let $\mathbf{X}_{G_k} = \bigoplus_{l=1}^{M_k}\{X^i | i \in C_l\}$ denote the states of all agents in the total set of clusters of the population. Hence $N = \sum_{l=1}^{M_k} |C_l|$. All spatially distributed clusters lie at the nodes of the graph G_k and interact via the

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weighted averages (1) defined by the finite graph G_k. For each agent $\mathcal{A}_i$, whose cluster is denoted by $C(i)$, the coupling term (also called the local graphon weighted mean field term) governing its interaction with other players via the network is given by:

$$Z_t^{i,G_k} = \frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{C(i),l}\frac{1}{|C_l|}\sum_{j\in C_l}\chi(X_t^j), \quad \forall t\in[0,T]. \quad (1)$$

The specification of the χ-modified mean field $\{Z_t^{i,G_k}; t\in[0,T]\}$ at the node G_k relies on the local modified mean fields $\chi(X_t^j)$ and the sectional information $g^k_{i,\bullet}$ of $\mathcal{A}_i$. Here all the individuals residing in the same cluster C_l, including the agent's local cluster $C(i)$, are symmetric and their average generates an overall impact on each agent $\mathcal{A}_i$ in the ith cluster via the local graphon weighted mean field term as shown in (2) below.

The state evolution of the collection of N agents $\mathcal{A}_i, 1\le i\le N<\infty$, is specified by a set of N control affine stochastic differential equations (SDEs) over a finite horizon of duration $T, 0<T<\infty$. For each agent $\mathcal{A}_i$, at some node the state evolution is given by

$$dX_t^i = (a(X_t^i) + bu_t^i + c(X_t^i)Z_t^{i,G_k})dt + \sigma dW_t^i, \quad \forall t\in[0,T], \quad (2)$$

where $a(\cdot), c(\cdot)$ are continuously differentiable bounded functions, and $\sigma>0$. These conditions will be strengthened as needed below to obtain the main results. Here $X_t^i\in\mathbb{R}$ denotes the state, $u_t^i\in\mathbb{R}$ the control input and Z_t^{i,G_k} the local graphon weighted mean field specified in (1). All initial states X_0^i are independent. Let $\{W^i, i=1,\cdots,N\}$ denote a collection of independent standard Brownian motions defined on a probability space $(\Omega,\mathbb{F},\mathbb{P})$ satisfying the usual conditions.

Furthermore, each agent $\mathcal{A}_i$ has a cost given by

$$J_i^N(u^i,u^{-i}) := \mathbb{E}\int_0^T\Big[\frac{r}{2}(u_t^i)^2 + \exp\Big(-\frac{q}{2}(X_t^i-\gamma(t)Z_t^{i,G_k})^{-2}\Big)\Big]dt, \quad (3)$$

where $1\le i\le N$, $\gamma(t)$ is a continuous function of time and u^{-i} denotes the controls of all agents other than $\mathcal{A}_i$. The two parameters r and q are positive. We note that the exponentiated negative inverse quadratic (ENIQ) running cost function on the system state in (3) vanishes at the origin and is strictly positive, monotonically increasing, infinitely differentiable and bounded by unity on $(0,\infty)$.

The configuration above constitutes a large scale dynamic stochastic game. A fundamental notion of a solution for these games is the Nash equilibrium, which is recalled in the following definition.

Any collection of controls for the large dynamic stochastic network games denoted $(u^{i*}, i=1,\cdots,N)$, is a *Nash equilibrium* if and only if, any unilateral deviation, from u^{i*} to any other control u^i, yields a higher cost. That is,

$$J_i^N(u^{i*},u^{-i*}) \le J_i^N(u^i,u^{-i*}),\ \forall i=1,\cdots,N. \quad (4)$$

Finding a Nash equilibrium for even a single cluster in a large model of the type as specified by (2)–(3) would generally be intractable but for the infinite population limit problem the theory of Mean Field Games ([14], [19]) provides an established approach (see [8]). The associated ϵ-Nash equilibrium results then yield approximate solutions to the original large finite population problems.

For non-uniform networks, different formulations have been given to this problem (see e.g. [3], [4], [17], [9], [21], [7]) and in the present paper we follow the Graphon Mean Field Games paradigm ([3], [4]).

In the large scale limit defined here, the number of nodes, M_k, of G_k tends to infinity and the smallest size of clusters at each node, $\min_{l=1:M_k}|C_l|$, tends to infinity, and hence the number of agents, N, also goes to infinity.

For simplicity of analysis we shall assume the limit measures have distribution functions which possess continuously differentiable densities, and, as a general notation for such graphon densities, we write

$$g:[0,1]^2\longrightarrow[0,\infty)$$
$$(\alpha,\beta)\mapsto g(\alpha,\beta).$$

We provide an example in which one considers a sequence of uniform attachment graphs [20], and obtains the following graphon density (in the limit)

$$g:[0,1]\times[0,1]\longrightarrow[0,1]$$
$$(\alpha,\beta)\mapsto g(\alpha,\beta)=1-\max\{\alpha,\beta\},$$

as illustrated in the figure below

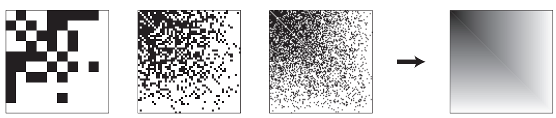

Figure 1: Graph Sequence Converging to its Limit [20]

Parallel to the standard MFG formulation, the infinite population of agents at all graphon nodes, $\alpha\in[0,1]$, admits representative agents, whose state evolution is given by control affine SDEs as the limiting form of (2) above:

$$dX_t^\alpha = (a(X_t^\alpha) + bu_t^\alpha + c(X_t^\alpha)Z_t^{\alpha,g})dt + \sigma dW_t^\alpha, \quad \forall t \in [0,T], \quad \forall \alpha \in [0,1], \tag{5}$$

where $(W_t^\alpha)_{t\in[0,T]}$ is a standard Brownian motion and $\sigma > 0$ is the noise intensity. The initial state X_0^α has probability distribution $\mu^\alpha(0,dx)$. Note that no form of stochastic process along the interval $\{\alpha \in [0,1]\}$ is defined in this paper.

In the large scale limit, each representative agent indexed by $\alpha \in [0,1]$ minimizes a cost function given by

$$J(u^\alpha,\mu) := \mathbb{E}\int_0^T \Big[\frac{r}{2}(u_t^\alpha)^2 + \exp\Big(-\frac{q}{2}(X_t^\alpha - \gamma(t)Z_t^{\alpha,g})^{-2}\Big)\Big]dt, \tag{6}$$

and at all nodes $\alpha \in [0,1]$, the global mean field term denoted $Z_t^{\alpha,g}, t \in [0,T]$, is defined as

$$Z_t^{\alpha,g} := \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)\mu^\beta(t,dx)d\beta, \tag{7}$$

where $\chi(x)$ is a bounded, integrable function of $x \in \mathbb{R}$ and $\mu^\beta(t,dx)$ is the distribution of X_t^β.

2. The Control Affine Exponentiated Costs GMFG Equations

In this section, we formalize and describe the solvability of the Graphon Mean Field Games associated with the control affine exponential costs model introduced in the previous section.

2.1. Formulation of the GMFG Problem

Define the following admissible control space,

$$\mathbb{A} := \{u : \Omega \times [0,T] \mapsto \mathbb{R} \mid u(\cdot) \text{ is } \mathbb{F}-\text{progressively measurable and } \mathbb{E}[\textstyle\int_0^T |u(t)|^2 dt] < \infty\},$$

and the corresponding instance of a Control Affine (Quadratic Gaussian) Graphon Mean Field Game (CA-GMFG) problem.

Find a two-parameter family of probability measures in $\mathcal{P}_2(\mathbb{R})$, denoted $\mu(\alpha,t)$, $\forall t \in [0,T]$, $\forall \alpha \in [0,1]$, such that:

1) **Agents' Control Problems**:
There exists α-nodal optimal control laws, denoted $u^{\alpha,o} := (u_t^{\alpha,o})_{t\in[0,T]} \in \mathbb{A}$ for all $\alpha \in [0,1]$, such that

$$J(u^{\alpha,o},\mu) = \min_{u^\alpha\in\mathbb{A}} J(u^\alpha,\mu) = \min_{u^\alpha\in\mathbb{A}} \mathbb{E}\int_0^T \Big[\frac{r}{2}(u_t^\alpha)^2 + \exp\Big(-\frac{q}{2}(X_t^\alpha - \gamma(t)Z_t^{\alpha,g})^{-2}\Big)\Big]dt \tag{8}$$

subject to the dynamics for all $t \in [0,T]$

$$dX_t^\alpha = \sigma dW_t^\alpha, + (a(X_t^\alpha) + bu_t^\alpha + c(X_t^\alpha)Z_t^{\alpha,g})dt \tag{9}$$

$$Z_t^{\alpha,g} = \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)\mu^\beta(t,dx)d\beta, \tag{10}$$

where $\mu^\beta(t,dx)$ is the distribution of X_t^β.

2) **Consistency Conditions**:
The optimal state trajectories $(X_t^{\alpha,\mu,o})_{t\in[0,T]}, \forall\alpha \in [0,1]$, generated in Part 1) satisfy the GMFG McKean-Vlasov consistency conditions:

$$\mu(\alpha,t) = \mathcal{L}(X_t^{\alpha,\mu,o}), \quad \forall(\alpha,t) \in [0,1]\times[0,T]. \tag{11}$$

2.2. Solvability of the Control Affine-GMFG Problem

The analysis in this section establishes that one can solve the Control Affine GMFG problem via the resolution of a system of Forward Backward Partial Differential Equations (FBPDEs) describing the value function and the probability density function of agents involved in the Control Affine GMFG problem.

We proceed in a two step approach. Firstly, by fixing probability density functions for the states of the representative agents we derive the Hamilton-Jacobi-Bellman (HJB) equations for their value functions together with the terminal conditions. Secondly, given the resulting control laws for the representative agents, we derive the Fokker-Planck-Kolmogorov (FPK) equations for their probability density functions together with initial conditions. Subject to the consistency condition on the generated density functions, these two coupled sets of equations constitute the entire Controlled Affine GMFG system.

HJB Equations

We introduce, for all $(\alpha,t,x) \in [0,1]\times[0,T]\times\mathbb{R}$ the probability density functions $p(\alpha,t,x)$ satisfying the condition

$$d\mu(\alpha,t)(x) = p(\alpha,t,x)dx,$$

and we define the systems' Hamiltonians in terms of the notation introduced above, namely,

$$\begin{aligned}
H\Big[t, x, &\frac{\partial V(\alpha,t,x)}{\partial x}, Z, u\Big] \\
&:= (a(x) + bu + c(x)Z)\frac{\partial V(\alpha,t,x)}{\partial x} \\
&+ \Big[\frac{r}{2}u^2 + \exp\Big(-\tfrac{q}{2}(x-\gamma(t)Z)^{-2}\Big)\Big], \qquad (12)
\end{aligned}$$

with $x, u, q, Z \in \mathbb{R}$, $\gamma(\cdot) \in C([0,T])$, and $V(\alpha,t,x)$ the value functions of the representative agents. Applying the dynamic programming principle, we obtain that the value functions are given as solutions to the HJB equations

$$\begin{aligned}
-\frac{\partial V(\alpha,t,x)}{\partial t} &= \inf_{u\in\mathbb{A}} H\Big[t, x, \frac{\partial V(\alpha,t,x)}{\partial x}, Z_t^{\alpha,g}, u\Big] \\
&\quad + \frac{\sigma^2}{2}\frac{\partial^2 V(\alpha,t,x)}{\partial x^2}, \\
&= \Big[\exp\Big(-\frac{q}{2}(x-\gamma(t)Z_t^{\alpha,g})^{-2}\Big) - \frac{b^2}{2r}\Big(\frac{\partial V(\alpha,t,x)}{\partial x}\Big)^2 \\
&\quad + \Big(a(x) + c(x)Z_t^{\alpha,g}\Big)\Big(\frac{\partial V(\alpha,t,x)}{\partial x}\Big)\Big] \\
&\quad + \frac{\sigma^2}{2}\Big(\frac{\partial^2 V(\alpha,t,x)}{\partial x^2}\Big), \qquad (13) \\
V(\alpha,T,x) &= 0,
\end{aligned}$$

where $Z_t^{\alpha,g}$ is given by

$$Z_t^{\alpha,g} = \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)\mu^\beta(t,dx)d\beta.$$

FPK Equations

Given the value functions and probability density functions, $\{V(\alpha,t,x), p(\alpha,t,x), (\alpha,t,x) \in [0,1]\times[0,T]\times\mathbb{R}\}$, we obtain the following best response controls, $\{u_t^{\alpha,o},\ (\alpha,t) \in [0,1]\times[0,T]\}$, and the closed-loop states, $\{X_t^{\alpha,o},\ (\alpha,t) \in [0,1]\times[0,T]\}$, for the representative agents

$$\begin{aligned}
u_t^{\alpha,o} &= -\frac{b}{r}\frac{\partial V(\alpha,t,X_t^{\alpha,o})}{\partial x}, \qquad X_t^{\alpha,o} = \xi^\alpha, \\
dX_t^{\alpha,o} &= \sigma dW_t^\alpha \\
&\quad + \Big((a(X_t^{\alpha,o}) + c(X_t^{\alpha,o})Z_t^{\alpha,g} - \frac{b^2}{r}\frac{\partial V(\alpha,t,X_t^{\alpha,o})}{\partial x}\Big)dt,
\end{aligned}$$

and derive the following FPK equations for the probability density functions associated with the SDEs describing the closed-loop states,

$$\begin{aligned}
\frac{\partial p(\alpha,t,x)}{\partial t} &= -\frac{\partial}{\partial x}\Big[p(\alpha,t,x)\Big(a(x) + c(x)Z_t^{\alpha,g} \\
&\quad - \frac{b^2}{r}\frac{\partial V(\alpha,t,x)}{\partial x}\Big)\Big] + \frac{\sigma^2}{2}\frac{\partial^2 p(\alpha,t,x)}{\partial x^2}, \qquad (14)
\end{aligned}$$

where the initial condition $p^\alpha(x) := p(\alpha,0,x)$ is given.

The coupled FBPDEs (13) and (14) constitute the Control Affine GMFG equations and their solutions are given by

$$\{V(\alpha,t,x), p(\alpha,t,x),\ (\alpha,t,x) \in [0,1]\times[0,T]\times\mathbb{R}\}.$$

For existence analysis, we introduce the following assumptions.

(A1) The functions $a(x)$, $a_x(x)$, $c(x)$, and $c_x(x)$ are bounded continuous functions, and a_x, c_x are both in the Hölder space $C^\gamma(\mathbb{R})$ with Hölder exponent $\gamma \in (0,1)$.

(A2) The function $\gamma(t)$ is continuously differentiable on $[0,T]$.

(A3) The initial probability density function $p^\alpha(x)$ is continuous in $(\alpha,x) \in [0,1]\times\mathbb{R}$ and $p^\alpha(\cdot) \in C^{2+\gamma}(\mathbb{R})$.

(A4) χ is bounded, Lipschitz continuous (with Lipschitz constant $\mathrm{Lip}(\chi)$) and

$$\int_{\mathbb{R}} |\chi(x)|dx < \infty.$$

(A5) $g : [0,1]^2 \to [0,1]$ is measurable function, and g maps $C([0,1])$ to $C([0,1])$, i.e., given $h \in C([0,1])$, the mapping

$$\alpha \to \int_0^1 g(\alpha,\beta)h(\beta)d\beta, \quad \alpha \in [0,1]$$

is a continuous function defined on $[0,1]$.

We will seek the solution of the HJB-FPK equation system in a suitable Hölder space. For this purpose, we introduce related Hölder semi-norms and norms.

2.3. Notation

If the function $h(x)$ is defined on a set $Q \subset \mathbb{R}^n$, we denote the norm $|h|_{0;Q} = \sup_{x\in Q}|g(x)|$ and the Hölder semi-norm $[h]_{\gamma;Q} = \sup_{x,x'} |h(x) - h(x')|/|x - x'|^\gamma$ for $\gamma \in (0,1)$. If $f(t,x)$ is defined on the set $Q_T = [0,T]\times Q$, define the Hölder semi-norms (see [16])

$$[f]_{\gamma/2,\gamma;Q_T} = \sup_{(t,x),(s,y)\in Q_T} \frac{|f(t,x) - f(s,y)|}{(|t-s|^{1/2} + |x-y|)^\gamma},$$

and

$$[f]_{1+\gamma/2,2+\gamma,Q_T} = [f_t]_{\gamma/2,\gamma;Q_T} + \sum_{i,j}[f_{x_i x_j}]_{\gamma/2,\gamma;Q_T}.$$

Denote the Hölder norms

$$|h|_{\gamma;Q} = |h|_{0;Q} + [h]_{\gamma;Q},$$
$$|f|_{\gamma/2,\gamma;Q_T} = |f|_{0;Q_T} + [f]_{\gamma/2,\gamma;Q_T},$$
$$|f|_{1+\gamma/2,2+\gamma;Q_T} = |f|_{0;Q_T} + |f_t|_{0;Q_T} + \sum_i |f_{x_i}|_{0;Q_T} + \sum_{i,j} |f_{x_ix_j}|_{0;Q_T} + [f]_{1+\gamma/2,2+\gamma;Q_T}.$$

The subscript Q or Q_T in the norm/semi-norm may be omitted if it is clear from the context. The Hölder space $C^{\gamma/2,\gamma}(Q_T)$ (resp., $C^{1+\gamma/2,2+\gamma}(Q_T)$) consists of all functions with $|f|_{\gamma/2,\gamma;Q_T} < \infty$ (resp., $|f|_{1+\gamma/2,2+\gamma;Q_T} < \infty$). The Hölder space $C^{2+\gamma}(Q)$ is similarly defined with the norm $|h|_{2+\gamma;Q} = |f|_{0;Q} + \sum_i |f_{x_i}|_{0;Q} + \sum_{i,j} |f_{x_ix_j}|_{0;Q} + \sum_{i,j}[f_{x_ix_j}]_{\gamma;Q}$. We will solve the HJB equation (13) and the FPK equation (14) in the Hölder space $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$.

We start with an informal description of the solution procedure for the GMFG. We take a generic coupling term Z (as a function of (α, t), to be properly specified later) in place of $Z_t^{\alpha,g}$ and find a unique solution $V^\alpha = V(\alpha, t, x)$ of the HJB equation and the resulting best response control law which, in turn determines the closed-loop state dynamics (using Z^α in place of $Z^{\alpha,g}$), such that X_t^α has probability density function $p(\alpha, t, x)$. Subsequently the closed-loop system generates

$$Z_1^\alpha(t) = \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)p(\beta,t,x)dxd\beta, \tag{15}$$

which is written in the compact form

$$Z_1 = \Phi(Z)$$

using a naturally defined operator Φ. Note that the family of probability density functions $(p(\beta,t,x))_{\beta\in[0,1]}$ in (15) has been determined using Z. Hence the solution of the GMFG may be characterized by a fixed point equation

$$Z = \Phi(Z),$$

where Z is viewed as a function of (α, t).

We need to specify a set $\mathcal{Z}$ that the operator Φ acts on. Let $b_0 = b^2/r$. Following the notation in [13], we denote $C_T^* = \exp([|a_x|_0 + |c_x|_0 \cdot |\chi|_0]T)$, $\mathrm{Lip}_x(L) = \sup_{t\in[0,T];x\in\mathbb{R},|z|\le|\chi|_0} L_x(t,x,z)$, $C_1^* = \mathrm{Lip}_x(L)C_T^*T$, and $C_2^* = |a|_0 + 2b_0C_1^* + |c|_0 \cdot |\chi|_0$, $C_3^* = \mathrm{Lip}(\chi)(C_2^*T^{1-\gamma/2} + \sqrt{2}T^{(1-\gamma)/2})$. Now we are ready to specify the set $\mathcal{Z}$ consisting of all Z satisfying the two conditions: (i) Z a continuous function of (t,α) defined on $[0,T]\times[0,1]$; (ii)

$$|Z^\alpha(t)| \le |\chi|_0, \quad |Z^\alpha(t) - Z^\alpha(s)| \le C_3^*|t-s|^{\gamma/2}, \tag{16}$$
$$\forall t, s \in [0,T], \ \alpha \in [0,1].$$

On $\mathcal{Z}$ we define the metric

$$d(Z,\hat{Z}) = \sup_\alpha |Z^\alpha - \hat{Z}^\alpha|_{\gamma/2;[0,T]}.$$

It is straightforward to show that $(\mathcal{Z}, d)$ is a complete metric space.

Given $Z \in \mathcal{Z}$, we fix $\alpha \in [0,1]$ and view Z^α as a function of t to solve the HJB equation (13) to get a unique solution $V(\alpha,t,x)$, $\alpha\in[0,1]$ (see [13, Theorem 2.3] for details), and subsequently to obtain a unique solution $p(\alpha,t,x)$ from (14) (see [13, Proposition 2.1]). Now we define

$$Z_1^\alpha(t) = \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)p(\beta,t,x)dxd\beta,$$

which can be written in terms of an operator Φ:

$$Z_1 = \Phi(Z), \quad \text{for} \quad Z \in \mathcal{Z}.$$

Theorem 2.1. *[13, Theorem 4.1] Under Assumptions (A1), (A2), (A3), (A4) and (A5). The mapping Φ is from $\mathcal{Z}$ to $\mathcal{Z}$, and there exists a constant C_0 such that for all $Z, \hat{Z} \in \mathcal{Z}$, one has*

$$\sup_\alpha |Z_1^\alpha - \hat{Z}_1^\alpha|_{\gamma/2;[0,T]} \le C_0 \sup_\alpha |Z^\alpha - \hat{Z}^\alpha|_{\gamma/2;[0,T]},$$

where $\hat{Z}_1 = \Phi(\hat{Z})$.

The constant C_0 can be determined using the known functions and parameters in the model (5)-(6) (see [13] for details). Under the above assumptions, Theorem 2.1 ensures Φ to be a Lipschitz mapping, and to be a contraction under suitable conditions (for instance, $C_0 < 1$ holds when either $|c|_0 + |c_x|_0 + [c_x]_\gamma + b_0$ or $\sup_\alpha \int_0^1 |g(\alpha,\beta)|d\beta + \int_{\mathbb{R}} |\chi(x)|dx$ is sufficiently small; see Remark 4.4 in [13]).

Under a contraction condition, the next theorem obtains a unique solution pair (V, p) to the Control Affine GMFG equations (13) and (14), where V and p are each jointly continuous in all the variables (t, x, α), such that for each fixed α, both V^α and p^α are in $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$.

Theorem 2.2. *Suppose all assumptions in Theorem 2.1 holds with $C_0 < 1$. Then the GMFG equation system (13)–(14) has a unique solution (V^α, p^α) in $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})\times C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$, $\alpha\in[0,1]$.*

Proof. Step 1. Given $C_0 < 1$, the fixed point equation $Z = \Phi(Z)$ has a unique solution $Z \in \mathcal{Z}$ since $(\mathcal{Z}, d)$ is a complete metric space. For each α, taking $Z^{\alpha,g} = Z^\alpha$ in (13) and (14), we obtain a well defined solution pair

(V^α, p^α) in $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})\times C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$. This establishes existence.

Step 2. To show uniqueness, suppose $(\bar{V}^\alpha, \bar{p}^\alpha)$, $\alpha \in [0,1]$, is a solution to (13)–(14). Set

$$Z^\alpha(t) = \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)\bar{p}(\beta,t,x)dxd\beta.$$

Then by the definition of the operator Φ, Z is in fact the unique solution of $Z = \Phi(Z)$. So $(\bar{V}^\alpha, \bar{p}^\alpha)$ is necessarily equal to (V^α, p^α) determined in Step 1. This proves uniqueness. □

3. Critical Nodes for GMFGs

Recall that the global mean field, $Z_t^{\alpha,g}$, defined by

$$Z_t^{\alpha,g} := \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)\mu^\beta(t,dx)d\beta, \tag{17}$$

is an interaction term describing the influence of the limit network on the dynamics of the representative agents at each node $\alpha \in [0,1]$.

In this section, we consider the particular nodes at which the first derivative of the global graphon mean field with respect to $\alpha \in [0,1]$ vanishes, which we call mean critical nodes.

We introduce another assumption.

(A6) $g(\alpha,\beta)$ is differentiable with respect to $\alpha \in [0,1]$, with $|g_\alpha(\alpha,\beta)| \le C_g$ for some fixed constant C_g and all (α,β), and the function $g_\alpha(\cdot,\cdot)$ maps $C([0,1])$ to $C([0,T])$, i.e., given $h \in C([0,1])$, the mapping

$$\alpha \to \int_0^1 g_\alpha(\alpha,\beta)h(\beta)d\beta, \quad \alpha \in [0,1]$$

is a continuous function of $\alpha \in [0,1]$.

Note that we view $Z_t^{\alpha,g}$ as a function of $(\alpha,t) \in [0,1]\times[0,T]$.

Proposition 3.1. *Assume that assumptions (A1) through (A6) hold and that Φ has a unique fixed point in $\mathcal{Z}$. Then the partial derivative $\frac{\partial Z_t^{\alpha,g}}{\partial\alpha}$ is defined at each $\alpha \in [0,1]$ and is continuous in $(\alpha,t) \in [0,1]\times[0,T]$. Moreover, for each given α, $\frac{\partial Z_t^{\alpha,g}}{\partial\alpha}$ belongs to $C^{\gamma/2}([0,T])$.*

Proof. Under the above assumptions, the HJB-FPK equation system has a unique classical solution. Further, given (A6), the dominated convergence theorem ensures that

$$\frac{\partial}{\partial_\alpha} Z_t^{\alpha,g} = \int_0^1 \int_{\mathbb{R}} g_\alpha(\alpha,\beta)\chi(x)\mu^\beta(t,dx)d\beta =: \psi(\alpha,t). \tag{18}$$

We proceed to show continuity of $\frac{\partial Z_t^{\alpha,g}}{\partial\alpha}$. Fix (α,t). Taking $t' \in [0,T]$ and $\alpha' \in [0,1]$, we estimate

$$|\psi(\alpha,t) - \psi(\alpha',t')| \le |\psi(\alpha,t) - \psi(\alpha',t)| + |\psi(\alpha',t) - \psi(\alpha',t')|. \tag{19}$$

For an arbitrary $\epsilon > 0$, under assumption (A6) there exists $\delta > 0$ such that for all α' with $|\alpha - \alpha'| \le \delta$, one has

$$|\psi(\alpha,t) - \psi(\alpha',t)| \le \epsilon.$$

For the second term in the sum in (19), we have

$$|\psi(\alpha',t) - \psi(\alpha',t')| \le \int_0^1 g_\alpha(\alpha',\beta)E|\chi(X_t^\beta) - \chi(X_{t'}^\beta)| \le C_g \mathrm{Lip}(\chi) \sup_\beta E|X_t^\beta - X_{t'}^\beta|. \tag{20}$$

By boundedness of $|a(x)|$, $|c(x)|$ and $|\frac{\partial V(\alpha,t,x)}{\partial x}|$ (see the estimate of $\sup_{\alpha,t,x}|\frac{\partial V(\alpha,t,x)}{\partial x}|$ in [13, Theorem 2.3]), there exists $\delta_1 > 0$ such that for all t' with $|t-t'| \le \delta_1$, one has $\sup_\beta E|X_t^\beta - X_{t'}^\beta| \le \frac{1}{1+C_g\mathrm{Lip}(\chi)}\epsilon$. Therefore, it follows that

$$|\psi(\alpha,t) - \psi(\alpha',t')| \le 2\epsilon$$

provided that $|\alpha - \alpha'| \le \delta$ and $|t - t'| \le \delta_1$ hold.

To show Hölder continuity, recall that $p(\beta,t,x)$ is in $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$; then we use the method in [13, sec 4] to show $\sup_{t\neq s}|\psi(\alpha,t) - \psi(\alpha,s)|/|t-s|^{\gamma/2} < \infty$. This completes the proof of the proposition. □

Definition: Mean Critical Node A node $\lambda \in [0,1]$ is a *mean critical node* for a GMFG system if the following local mean field stationarity condition holds for $Z_t^{\alpha,g}$ at $\lambda \in [0,1]$,

$$\frac{\partial}{\partial\alpha} Z_t^{\alpha,g}\Big|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T]. \tag{21}$$

Definition: α-Nash Critical Node A node $\lambda \in [0,1]$ is an *α - Nash critical node* for a GMFG system if the following local Nash value stationary condition holds for $V_t^{\alpha,g}$ at $\lambda \in [0,1]$,

$$\frac{\partial}{\partial\alpha} V_t^{\alpha,g}\Big|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T]. \tag{22}$$

For three particular examples of graphons one can readily identify mean critical nodes and observe that they coincide with specific nodes in the family of graphs whose limits are associated with the graphons as follows:

E1 Consider first the limit graphon of a sequence of finite Erdös-Rényi graphs for which the graphon limit and associated local mean fields are, respectively:

$$g(\alpha,\beta) := k \in (0,1), \quad \forall(\alpha,\beta) \in [0,1]^2,$$

$$Z_t^{\alpha,g} = k \int_0^1 \mathbb{E}[X_t^{\beta,o}]d\beta, \ \forall(\alpha,t) \in [0,1] \times [0,T].$$

So for all $\lambda \in [0,1]$:

$$\frac{\partial}{\partial\alpha} Z_t^{\alpha,g}\Big|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T].$$

That is to say, for the graphon limit of Erdös-Rényi finite graphs all nodes $\lambda \in [0,1]$ are mean critical nodes for the associated Control Affine GMFG problem.

E2 Consider next the uniform attachment graphon:

$$g(\alpha,\beta) = 1 - \max\{\alpha,\beta\}, \ \forall(\alpha,\beta) \in [0,1]^2.$$

Then, we can compute that for all $(\alpha,t) \in [0,1] \times [0,T]$

$$\begin{aligned} Z_t^{\alpha,g} &= \int_0^1 (1 - \max\{\alpha,\beta\})\mathbb{E}[X_t^{\beta,o}]d\beta, \quad (23) \\ &= (1-\alpha)\int_0^\alpha \mathbb{E}[X_t^{\beta,o}]d\beta + \int_\alpha^1 (1-\beta)\mathbb{E}[X_t^{\beta,o}]d\beta, \end{aligned}$$

Differentiating with respect to the index α yields:

$$\frac{\partial}{\partial\alpha} Z_t^{\alpha,g} = -\int_0^\alpha \mathbb{E}[X_t^{\beta,o}]d\beta, \ \forall t \in [0,1]. \quad (24)$$

from which it follows that for $\lambda = 0 \in [0,1]$:

$$\frac{\partial}{\partial\alpha} Z_t^{\alpha,g}\Big|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T].$$

That is to say, for the uniform attachment graphon, the root node is a mean critical node.

E3 As a third example consider the negative exponentiated graphon function

$$g(\alpha,\beta) = \exp\Big(-(\alpha^m - \beta^m)\Big), \quad 2 \le m, \ m \in \mathbb{N}, \ \forall(\alpha,\beta) \in [0,1]^2.$$

Then, for all $(\alpha,t) \in [0,1] \times [0,T]$

$$Z_t^{\alpha,g} = \int_0^1 \exp\Big(-(\alpha^m - \beta^m)\Big)\mathbb{E}[X_t^{\beta,o}]d\beta,$$

and differentiating yields for all $t \in [0,1]$:

$$\frac{\partial}{\partial\alpha} Z_t^{\alpha,g} = -m\alpha^{m-1}\int_0^1 \exp\Big(-(\alpha^m - \beta^m)\Big)\mathbb{E}[X_t^{\beta,o}]d\beta,$$

and hence for $\lambda = 0$:

$$\frac{\partial}{\partial\alpha} Z_t^{\alpha,g}\Big|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T].$$

So for the negative exponential graphon functions with index greater than one, the node $\alpha = 0$ is a mean critical node.

These examples indicate that the structure of the networks modelled by graph limits play a key role in the interaction between agents in the associated GMFGs.

4. Stationarity Properties of the Value Functions

Proposition 4.1. *Suppose that all assumptions in Proposition 3.1 hold. Assume $c(x) = 0$ for all $x \in \mathbb{R}$, $\gamma(t) \neq 0$ for all $t \in [0,T]$, and that the solution to the Control Affine GMFG problem admits α-Nash critical nodes $\lambda \in [0,1]$, that is to say*

$$\frac{\partial V(\alpha,t,x)}{\partial\alpha}\Big|_{\alpha=\lambda} = 0, \ \forall(t,x) \in [0,T] \times \mathbb{R}. \quad (25)$$

Then these nodes are mean critical nodes for the Control Affine GMFG system; namely, at these nodes the local mean field is stationary:

$$\frac{\partial}{\partial\alpha} Z_t^{\alpha,g}\Big|_{\alpha=\lambda} = 0, \quad \forall t \in [0,T]. \quad (26)$$

Conversely, subject to the same conditions, mean field critical nodes are α-Nash critical nodes, i.e. nodes at which the Control Affine GMFG value function is stationary.

Proof. Suppose (25) holds. Differentiating the Control Affine GMFG equations (13)-(14) with respect to α yields the function

$$W(\alpha,t,x) := \frac{\partial V(\alpha,t,x)}{\partial\alpha}, \ \forall(\alpha,t,x) \in [0,1] \times [0,T] \times \mathbb{R},$$

as a solution to the PDE:

$$
\begin{aligned}
-\frac{\partial W(\alpha,t,x)}{\partial t} &= -q\gamma(t)\Big(x-\gamma(t)Z_t^{\alpha,g}\Big)^{-3} \\
&\times \exp\Big[-\frac{q}{2}\Big(x-\gamma(t)Z_t^{\alpha,g}\Big)^{-2}\Big]\frac{\partial}{\partial\alpha}\Big(Z_t^{\alpha,g}\Big) \\
&+ (a(x)+c(x)Z_t^{\alpha,g})\frac{\partial W(\alpha,t,x)}{\partial x} \\
&+ c(x)\frac{\partial}{\partial\alpha}\Big(Z_t^{\alpha,g}\Big)\frac{\partial V(\alpha,t,x)}{\partial x} \qquad (27)\\
&- \frac{b^2}{r}\frac{\partial V(\alpha,t,x)}{\partial x}\frac{\partial W(\alpha,t,x)}{\partial x} \\
&+ \frac{\sigma^2}{2}\frac{\partial^2 W(\alpha,t,x)}{\partial x^2}, \\
&\qquad (\alpha,t,x)\in[0,1]\times[0,T]\times\mathbb{R}
\end{aligned}
$$

with terminal conditions

$$W(\alpha,T,x)=0,\ \ (\alpha,x)\in[0,1]\times\mathbb{R}.$$

Once $V(\alpha,t,x)$ has been determined, equation (27) becomes a linear parabolic equation with coefficients from a Hölder space, where, in particular, $\frac{\partial Z_t^{\alpha,g}}{\partial\alpha}$ has Hölder continuity in t (see Proposition 3.1). By the theory of such equations (see [18, p. 320]), (27) has a unique classical solution W^α in $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$ for given α.

Recalling that $c(x)=0, x\in\mathbb{R}$, we see that at any given $\lambda\in[0,1]$ for which

$$W(\lambda,t,x)=0,\qquad (t,x)\in[0,T]\times\mathbb{R},$$

the PDE (27) for $W(\cdot,\cdot,\cdot)$ takes the form

$$
\begin{aligned}
0 = -q\gamma(t)\Big(x-\gamma(t)Z_t^{\lambda,g}\Big)^{-3}\exp\Big[-\frac{q}{2}\Big(x-\gamma(t)Z_t^{\lambda,g}\Big)^{-2}\Big] \\
\times\Big(\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\Big|_{\alpha=\lambda}\Big),\qquad (t,x)\in[0,T]\times\mathbb{R},\quad (28)
\end{aligned}
$$

and hence

$$\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\Big|_{\alpha=\lambda}=0,\quad t\in[0,T]. \qquad (29)$$

Consequently $\lambda\in[0,1]$ is a mean field critical node.

The converse implication of the proposition holds since the boundary condition for the $W(\cdot,\cdot,\cdot)$ function is

$$W(\alpha,T,x)=\frac{\partial V(\alpha,T,x)}{\partial\alpha}=0,\quad (\alpha,x)\in[0,1]\times\mathbb{R},$$

due to the boundary condition on the value function being $V(\alpha,T,x)=0, (\alpha,t,x)\in[0,1]\times[0,T]\times\mathbb{R}$.

But then setting

$$\frac{\partial}{\partial\alpha}Z_t^{\alpha,g}\Big|_{\alpha=\lambda}=0,\quad \forall t\in[0,T], \qquad (30)$$

in the PDE (27) for $W(\cdot,\cdot,\cdot)$ results in the unique solution satisfying

$$\frac{\partial V(\alpha,t,x)}{\partial\alpha}\Big|_{\alpha=\lambda}=W(\alpha,t,x)|_{\alpha=\lambda}=0,\ \forall(t,x)\in[0,T]\times\mathbb{R}, \qquad (31)$$

as required. □

This result shows that, under specific conditions, mean critical nodes can be readily identified as nodes at which the value functions are stationary. This result allows for the identification of mean critical nodes directly from the solutions to the Control Affine GMFG equations (13) and (14).

5. Conclusion

In this paper a class of Graphon Mean Field Games with control affine non-linear dynamics and exponentiated negative inverse quadratic (ENIQ) cost functions has been considered. Under a contraction condition, the existence and uniqueness of solutions to the relevant GMFG equations is established. It has been shown that a node at which the equilibrium Nash value is stationary with respect to location is such that the local mean field is also stationary with respect to location and conversely. In future work the analysis will be extended with analyses of the existence and uniqueness of solutions to those GMFG equations which arise within the following generalizations: (i) the class of systems where the dynamics of each agent are also an affine function of the local mean field, (ii) systems subject to different varieties of running costs, including quadratic and logistic, and (iii) those resulting from the influence of specified classes of graphon limits [1] in arbitrary finite dimensions.

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