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LQG Graphon Mean Field Games: Analysis via Graphon Invariant Subspaces

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***Abstract*—This paper studies approximate solutions to large-scale linear quadratic stochastic games with homogeneous nodal dynamics parameters and heterogeneous network couplings within the graphon mean field game framework in [2]–[4]. A graphon time-varying dynamical system model is first formulated to study the finite and then limit problems of linear quadratic Gaussian graphon mean field games (LQG-GMFG). The Nash equilibrium of the limit problem is then characterized by two coupled graphon time-varying dynamical systems. Sufficient conditions are established for the existence of a unique solution to the limit LQG-GMFG problem. For the computation of LQG-GMFG solutions, two methods are established and employed where one is based on fixed point iterations and the other on a decoupling operator Riccati equation; furthermore, two corresponding sets of solutions are established based on spectral decompositions. Finally, a set of numerical simulations on networks associated with different types of graphons are presented.**

***Index Terms*—Large-scale networks, mean field games, complex networks, graphon control, infinite dimensional systems.**

I. Introduction

Many applications such as social networks, epidemic networks, market networks, communication networks and smart grids involve strategic decisions over a large number of agents coupled via large-scale heterogeneous network structures. The large cardinalities of the underlying networks and the complexity of the underlying network couplings in dynamics and decision strategies make such problems challenging or even intractable to analyze by standard methods. To characterize large graphs and study the convergence of (dense) graph sequences to their limits, graphon theory is established in the combinatorics and computer science communities [5]–[8]. It provides a theoretical tool to model large networks, network limits and networks with uncertainties [5]–[7], and has been applied to study various problems on large networks such as dynamical systems [9]–[12], network centrality [13], random walks [14], signal processing [15], graph neural networks [16], epidemic models ([17], [18]), control of very large-scale networks ([17], [19]–[24]), and static ([25], [26]) and dynamic game problems on graphons ([2]–[4], [27]).

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Game theoretic models with various interpretations of the underlying networks have been developed by various authors (see e.g. [28]–[31]) to study strategic decision problems for finite populations on finite networks. One challenging aspect in these problems is that when the underlying network becomes very large, the solution becomes extremely complex or even computationally intractable. To model and solve dynamic game problems on large-scale non-uniform networks with a large number of agents, Graphon Mean Field Game (GMFG) theory was proposed and developed in [2]–[4], which generalizes the classical mean field game theory in the sense that each node may be influenced by a different local mean field. Mean field game problems with non-uniform cost couplings were studied in an earlier paper [32], and mean field game problems on graphs with different interpretations of the underlying graphs have also been treated in [33]–[35]. In [33], [34], the graph represents physical constraints on the state space of the mean field game problems. In [35], linear quadratic mean field games over Erdös-Rényi graphs are studied where the associated asymptotic game is a classical mean field game. Recent works on mean field game problems on networks include [36], [37].

There are two classes of closely related mean field game problems on networks in the papers above depending on the definitions of nodes: (i) networks of mean field (or measure) couplings where each node on the network represents a population [2]–[4]; (ii) networks of individual state couplings where each node represents an agent (see for instance [27], [32], [35]–[37]). In the current paper, each node represents a population of homogeneous agents. The idea of associating each node with a population of agents has been employed in various application contexts, such as community structures on social networks [38], epidemics modelling [39], and neuronal dynamics [40], among others, which explores agent similarities to simplify the analysis of large-scale network problems.

The LQG-GMFG solution method in this current paper is as follows: first we identify the limit system when the size of the local nodal population and the size of the graph go to infinity; the Nash equilibrium for the limit system is then characterized by two coupled (global) graphon dynamical system equations; finally, each agent can then identify an approximated Nash strategy for the original LQG dynamic games on networks following the Nash equilibrium for the limit system.

The main contributions of this paper include:

- the characterization of the solution to the limit LQG-GMFG problem by two coupled (global) graphon time-varying dynamical systems;

- the establishment of sufficient conditions on the existence of a unique solution to the limit LQG-GMFG problem;
- the development of two spectral-based solution methods for solving the limit LQG-GMFG problems, one based on fixed point iterations and the other based on a decoupling operator Riccati equation.

Notation: $\mathbb{R}$ denotes the set of real numbers. Bold face letters (e.g. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{u}$) are used to represent graphons, compact operators and functions. Blackboard bold letters (e.g. $\mathbb{A}$, $\mathbb{B}$) are used to denote linear operators which are not necessarily compact. $\mathbb{A}^\intercal$ denotes the adjoint operator of $\mathbb{A}$. $\mathcal{W}_c$ denotes the set of all bounded symmetric measurable functions $\mathbf{M} : [0,1]^2 \to [-c,c]$ with $c>0$; $\mathcal{W}_0$ denotes the set of all bounded symmetric measurable functions $\mathbf{M} : [0,1]^2 \to [0,1]$. For a Hilbert space $\mathcal{H}$, $\mathcal{L}(\mathcal{H})$ denotes the Banach algebra of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. $\mathcal{L}(\mathcal{H})$ endowed with the uniform operator topology is denoted by $\mathcal{L}_u(\mathcal{H})$. For a Banach space $\mathcal{X}$, $C([0,T];\mathcal{X})$ denotes the set of continuous functions from $[0,T]$ to $\mathcal{X}$. Let $\oplus$ denote direct sum. Let $\otimes$ denote matrix Kronecker product. For any matrix $Q\in\mathbb{R}^{n\times n}$, $Q\geq 0$ (resp. $Q>0$) means $Q^\intercal = Q$ and $x^\intercal Q x \geq 0$ (resp. $x^\intercal Q x > 0$) for all $x\in\mathbb{R}^n$. For $x\in\mathbb{R}^n$, $Q\in\mathbb{R}^{n\times n}$ and $Q\geq 0$, let $\|x\|_Q^2 \triangleq x^\intercal Q x$. Let $(L^2[0,1])^n \triangleq \underbrace{L^2[0,1]\times\cdots\times L^2[0,1]}_{n}$. The inner product in $(L^2[0,1])^n$ is defined as follows: for $\mathbf{v},\mathbf{u} \in (L^2[0,1])^n$, $\langle \mathbf{u},\mathbf{v}\rangle \triangleq \sum_{i=1}^n \int_{[0,1]} \mathbf{v}_i(\alpha)\mathbf{u}_i(\alpha)d\alpha = \int_{[0,1]}\langle \mathbf{v}(\alpha),\mathbf{u}(\alpha)\rangle_{\mathbb{R}^n} d\alpha$ where $\mathbf{u}_i(\cdot) \in L^2[0,1]$ with $i\in\{1,\ldots,n\}$ denotes the ith component of $\mathbf{u}$ and $\mathbf{u}(\alpha)\in\mathbb{R}^n$ denotes the vector associated with index $\alpha\in[0,1]$. The space $(L^2[0,1])^n$ with the above inner product is a Hilbert space with the corresponding norm $\|\mathbf{v}\|_2 \triangleq \left(\int_{[0,1]}\|\mathbf{v}(\alpha)\|^2_{\mathbb{R}^n} d\alpha\right)^{\frac{1}{2}}$. We use $L^2([0,T];(L^2[0,1])^n)$ to denote the Hilbert space of equivalence classes of strongly measurable (in the Böchner sense [41, p.103]) mappings from $[0,T]$ to $(L^2[0,1])^n$ that are integrable with the norm $\|\mathbf{x}\|_{L^2([0,T];(L^2[0,1])^n)} = \left(\int_0^T \|\mathbf{x}(t)\|_2^2 dt\right)^{\frac{1}{2}}$. With an abuse of notation, let $\|\cdot\|_{\text{op}}$ denote the operator norm for both $\mathcal{L}((L^2[0,1])^n)$ and $\mathcal{L}(L^2[0,1])$, as it will become clear in the specific context which operator norm is referred to. The function $\mathbf{1}\in L^2[0,1]$ is defined as follows: for all $\alpha\in[0,1]$, $\mathbf{1}(\alpha)=1$. For any vector $v\in\mathbb{R}^n$, $v\mathbf{1}$ denotes the function in $(L^2[0,1])^n$ such that $(v\mathbf{1})(\alpha)=v$ for all $\alpha\in[0,1]$. $\mathbf{1}_n$ denotes the n-dimensional vector of ones and I_N denotes the identity matrix of dimension $N\times N$.

II. Preliminaries

A. Graphs, Graphons and Graphon Operators

A graph $G=(V,E)$ is specified by a node set $V=\{1,2,\ldots,N\}$ and an edge set $E\subset V\times V$. The corresponding adjacency matrix $M=[m_{ij}]$ is defined as follows: $m_{ij}=1$ if $(i,j)\in E$; otherwise $m_{ij}=0$. A graph is undirected if its edge pair is unordered. For a weighted undirected graph, m_{ij} in its adjacency matrix M is given by the weight between nodes i and j. Furthermore, an adjacency matrix can be represented as a pixel diagram on the unit square $[0,1]^2\subset\mathbb{R}^2$, which corresponds to a graphon step function [8] (see Fig. 1).

Graphons are defined as bounded symmetric Lebesgue measurable functions $\mathbf{M}:[0,1]^2\to[0,1]$. The space of graphons endowed with the *cut metric* (see [8]) allows the definition of the convergence of graph sequences. In this paper, we consider bounded symmetric Lebesgue measurable functions $\mathbf{M}:[0,1]^2\to[-c,c]$ with $c>0$, and the space of all such functions is denoted by $\mathcal{W}_c$. The space $\mathcal{W}_c$ is compact under the cut metric after identifying equivalent points of cut distance zero [8].

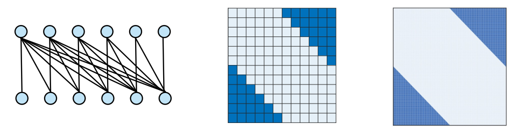

Fig. 1: A half graph [8], its pixel diagram, and its limit graphon

A graphon $\mathbf{M}\in\mathcal{W}_c$ also defines a self-adjoint bounded linear operator from $L^2[0,1]$ to $L^2[0,1]$ as follows:

$$(\mathbf{M}\mathbf{u})(\alpha) = \int_{[0,1]} \mathbf{M}(\alpha,\eta)\mathbf{u}(\eta)d\eta, \quad \forall\alpha\in[0,1], \tag{1}$$

where $\mathbf{u}$, $\mathbf{M}\mathbf{u}\in L^2[0,1]$. Moreover, graphons can be associated with operators from $(L^2[0,1])^n$ to $(L^2[0,1])^n$. Let $\mathcal{L}\left(\left(L^2[0,1]\right)^n\right)$ represent the set of bounded linear operators from $\left(L^2[0,1]\right)^n$ to $\left(L^2[0,1]\right)^n$. For any general bounded linear operator $\mathbb{T}\in\mathcal{L}\left(L^2[0,1]\right)$ and $D\in\mathbb{R}^{n\times n}$, the operator $[D\mathbb{T}]\in\mathcal{L}\left((L^2[0,1])^n\right)$ is defined as follows: for any $\mathbf{v}\in\left(L^2[0,1]\right)^n$ and any index $\alpha\in[0,1]$,

$$([D\mathbb{T}]\mathbf{v})(\alpha) \triangleq D\begin{pmatrix}(\mathbb{T}\mathbf{v}_1)(\alpha)\\ \vdots \\ (\mathbb{T}\mathbf{v}_n)(\alpha)\end{pmatrix}\in\mathbb{R}^n, \tag{2}$$

where $\mathbf{v}_i\in L^2[0,1]$ denotes the ith component of $\mathbf{v}\in(L^2[0,1])^n$. We use the square bracket $[\cdot]$ in (2) to indicate that the operator is in $\mathcal{L}\left(\left(L^2[0,1]\right)^n\right)$. The kth ($k\geq 0$) power function of $[D\mathbb{T}]$ is given by $[D\mathbb{T}]^k=[D^k\mathbb{T}^k]$, where $\mathbb{T}^0$ is formally defined as the identity operator from $L^2[0,1]$ to $L^2[0,1]$. Following (2), the operator $[D\mathbf{M}]\in\mathcal{L}\left(\left(L^2[0,1]\right)^n\right)$ with $D\in\mathbb{R}^{n\times n}$ and $\mathbf{M}\in\mathcal{W}_c$ is therefore defined as follows: for any $\mathbf{v}\in\left(L^2[0,1]\right)^n$ and any index $\alpha\in[0,1]$,

$$([D\mathbf{M}]\mathbf{v})(\alpha) \triangleq D\begin{pmatrix}(\mathbf{M}\mathbf{v}_1)(\alpha)\\ \vdots \\ (\mathbf{M}\mathbf{v}_n)(\alpha)\end{pmatrix}\in\mathbb{R}^n. \tag{3}$$

Since $[D\mathbf{M}]$ is a bounded linear operator from $\left(L^2([0,1])\right)^n$ to $\left(L^2([0,1])\right)^n$, it generates a uniformly continuous (hence strongly continuous) semigroup [42] given by $S_{[D\mathbf{M}]}(t) = \exp(t[D\mathbf{M}]) \triangleq \sum_{k=0}^\infty \frac{1}{k!}t^k[D\mathbf{M}]^k$, $t\geq 0$. Following the definition in (2), for the identity operator $\mathbb{I}\in\mathcal{L}\left(L^2[0,1]\right)$ and $D\in\mathbb{R}^{n\times n}$, the operation $[D\mathbb{I}]$ satisfies the following: for any $\mathbf{v}\in\left(L^2[0,1]\right)^n$ and any index $\alpha\in[0,1]$,

$$([D\mathbb{I}]\mathbf{v})(\alpha) \triangleq D\left(\mathbf{v}_1(\alpha)\ \ldots\ \mathbf{v}_n(\alpha)\right)^\intercal = D\mathbf{v}(\alpha)\in\mathbb{R}^n.$$

B. Invariant Subspace and Component-Wise Decomposition

Let $\mathcal{H}$ denote a Hilbert space. An *invariant subspace* of a bounded linear operator $\mathbb{T} \in \mathcal{L}(\mathcal{H})$ is defined as any subspace $\mathcal{S}_{\mathcal{H}} \subset \mathcal{H}$ such that $\mathbb{T}\mathcal{S}_{\mathcal{H}} \subset \mathcal{S}_{\mathcal{H}}$. Then the subspace $\mathcal{S}_{\mathcal{H}}$ is called $\mathbb{T}$-*invariant*. Since a graphon $\mathbf{M} \in \mathcal{W}_c$ defines a self-adjoint operator as in (1), for any invariant subspace $\mathcal{S} \subset L^2[0,1]$ of $\mathbf{M}$, $\mathcal{S}^{\perp}$ is also an invariant subspace of $\mathbf{M}$ (see [24]) where $S^{\perp}$ denotes the orthogonal complement subspace of $\mathcal{S}$ in $L^2[0,1]$. Let $(\mathcal{S})^n \triangleq \underbrace{\mathcal{S} \times \ldots \times \mathcal{S}}_{n} \subset (L^2[0,1])^n$. Clearly, by definition, $(\mathcal{S} \oplus \mathcal{S}^{\perp})^n = (L^2[0,1])^n$. Any $\mathbf{v} \in (L^2[0,1])^n$ can be uniquely decomposed through its components as

$$\mathbf{v}_i = \bar{\mathbf{v}}_i + \mathbf{v}_i^{\perp}, \quad \forall i \in \{1, ..., n\} \tag{4}$$

where $\bar{\mathbf{v}}_i \in \mathcal{S} \subset L^2[0,1]$ and $\mathbf{v}_i^{\perp} \in \mathcal{S}^{\perp} \subset L^2[0,1]$. We call the decomposition in (4) the *component-wise decomposition* of $\mathbf{v}$ into $(\mathcal{S})^n$ and $(\mathcal{S}^{\perp})^n$, and denote it by $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}^{\perp}$ where $\bar{\mathbf{v}} \in (\mathcal{S})^n$ and $\mathbf{v}^{\perp} \in (\mathcal{S}^{\perp})^n$ (see [24]). Clearly, any eigenspace of $\mathbf{M}$ (that is, the space spanned by some eigenfunctions) is an invariant subspace of $\mathbf{M}$. For any $\mathbf{v} \in (L^2[0,1])^n$, we have a unique component-wise decomposition as (4) into a given eigenspace and the associated orthogonal complement. The *kernel* (or *nullspace*) of $\mathbf{M} \in \mathcal{W}_c$ is denoted as $\ker(\mathbf{M}) \triangleq \big\{\mathbf{v} \in L^2[0,1] : \mathbf{M}\mathbf{v} = \mathbf{0}\big\}$. By its definition, $\ker(\mathbf{M})$ is an invariant subspace of $\mathbf{M} \in \mathcal{W}_c$. The *characterizing graphon invariant subspace* of $\mathbf{M} \in \mathcal{W}_c$ is defined as the subspace $\mathcal{S}_* \subset L^2[0,1]$ such that $\mathcal{S}_* = \ker(\mathbf{M})^{\perp}$.

III. Graphon Dynamical Systems

A. Graphon Dynamical System Model

Consider the graphon time-varying dynamical system

$$\dot{\mathbf{x}}(t) = [A(t)\mathbb{I} + D(t)\mathbf{M}]\mathbf{x}(t) + [B(t)\mathbb{I} + E(t)\mathbf{M}]\mathbf{u}(t) \tag{5}$$

where $\mathbf{M} \in \mathcal{W}_c$, $\mathbf{x}(t) \in \big(L^2[0,1]\big)^n$ for each $t \in [0,T]$. The admissible control $\mathbf{u}(\cdot)$ lies in $L^2([0,T]; (L^2[0,1])^n)$. Moreover, $A(\cdot)$, $B(\cdot)$, $D(\cdot)$ and $E(\cdot)$ are assumed to be continuous from $[0,T]$ to $\mathbb{R}^{n\times n}$. Let $\mathbb{A}_{\circ}(t) = [A(t)\mathbb{I} + D(t)\mathbf{M}]$ and $\mathbb{B}(t) = [B(t)\mathbb{I} + E(t)\mathbf{M}]$. A *mild solution* of (5) is defined as the solution $\mathbf{x}$ that is continuous over $[0,T]$ and satisfies the integral equation

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \left(\mathbb{A}_{\circ}(\tau)\mathbf{x}(\tau) + \mathbb{B}(\tau)\mathbf{u}(\tau)\right) d\tau. \tag{6}$$

Consider the initial value problem

$$\dot{\mathbf{x}}(t) = \mathbb{A}_{\circ}(t)\mathbf{x}(t), \quad \mathbf{x}(s) = \mathbf{x}_s, \quad 0 \le s \le t \le T, \tag{7}$$

where $\mathbb{A}_{\circ}(t) = [A(t)\mathbb{I} + D(t)\mathbf{M}]$. Clearly $\mathbb{A}_{\circ}(t)$ for every $t \in [0,T]$ is bounded and continuous under the uniform operator topology. Hence the classical solution[1] to (7) exists and is unique ([42, Thm. 5.1, Ch. 5]).

Definition 1 (Evolution Operator) The evolution operator associated with $\mathbb{A}_{\circ}(\cdot)$ is defined as the two-parameter family of operators $\Phi(\cdot,\cdot)$ that satisfies $\Phi(t,s)\mathbf{x}_s = \mathbf{x}(t)$, for $0 \le s \le t \le T$ where $\mathbf{x}$ denotes the classical solution to (7). □

We note that $\mathbb{A}_{\circ}(t) = [A(t)\mathbb{I} + D(t)\mathbf{M}]$ is a bounded linear operator from $(L^2[0,1])^n$ to $(L^2[0,1])^n$ and $\mathbb{A}(\cdot)$ is continuous under the uniform operator topology. Hence following [42, Thm. 5.2, Ch. 5], the evolution operator $\Phi(\cdot,\cdot)$ satisfies

$$\frac{\partial \Phi(t,\tau)}{\partial t} = \mathbb{A}_{\circ}(t)\Phi(t,\tau), \;\; \Phi(\tau,\tau) = \mathbb{I}, \;\; t,\tau \in [0,T], \tag{8}$$

in $\mathcal{L}_u\big((L^2[0,1])^n\big)$ (the space of all bounded linear operators on $(L^2[0,1])^n$ under the uniform operator topology).

Lemma 1 (Mild Solution) *The system* (5) *has a unique mild solution* $\mathbf{x}$ *in* $C([0,T]; (L^2[0,1])^n)$ *given by*

$$\mathbf{x}(t) = \Phi(t,0)\mathbf{x}(0) + \int_0^t \Phi(t,\tau)[B(\tau)\mathbb{I} + D(\tau)\mathbf{M}]\mathbf{u}(\tau)d\tau \tag{9}$$

with $\Phi(\cdot,\cdot)$ *as the evolution operator associated with* $[A(\cdot)\mathbb{I} + D(\cdot)\mathbf{M}]$.

Proof Since $A(\cdot)$, $B(\cdot)$ and $D(\cdot)$ are continuous functions from $[0,T]$ to $\mathbb{R}^{n\times n}$, we obtain that for any $\mathbf{v} \in (L^2[0,1])^n$, $\big[A(\cdot)\mathbb{I} + D(\cdot)\mathbf{M}\big]\mathbf{v}$ and $\big[B(\cdot)\mathbb{I} + D(\cdot)\mathbf{M}\big]\mathbf{v}$ are continuous functions from $[0,T]$ to $(L^2[0,1])^n$. By the Uniform Boundedness Principle, there exists $c > 0$ such that $\big\|\big[B(t)\mathbb{I} + D(t)\mathbf{M}\big]\big\|_{\text{op}} \le c$, for all $t \in [0,T]$. This together with $\mathbf{u}(\cdot) \in L^2([0,T]; (L^2[0,1])^n)$ implies

$$\Big(\big[B(t)\mathbb{I} + D(t)\mathbf{M}\big]\mathbf{u}(t)\Big)_{t\in[0,T]} \in L^2([0,T]; (L^2[0,1])^n),$$

that is, it is Böchner measurable. Furthermore, we note that $(L^2[0,1])^n$ is a reflexive Banach space. Therefore, all the conditions in [43, Lem. 3.2, Prop. 3.4, Prop. 3.6, Part II] are verified and we obtain that the system (5) is well defined and has a unique mild solution and the solution is given by (9).■

Lemma 2 (Classical Solution) *If* $\mathbf{u} \in C([0,T]; (L^2[0,1])^n)$*, then the system* (5) *has a unique classical solution*[2] $\mathbf{x}$*, and the classical solution is also given by* (9)*.*

Proof The proof follows that of [42, Thm. 5.1, Ch. 5]. First by Picard's iterations, one can establish the existence of a unique mild solution satisfying (6) based on the observation that $\mathbb{A}_{\circ}(\cdot)$ is continuous in the uniform operator topology. Then the fact that $\mathbf{x}$ lies in $C([0,T]; (L^2[0,1])^n)$ together with the assumption $\mathbf{u} \in C([0,T]; (L^2[0,1])^n)$ implies that the right-hand side of (6) is differentiable. By differentiating both sides of (6) in the classical sense, we obtain the classical solution to (5). Clearly, from the analysis above, the unique classical solution is also given by (9) (see also [42, p. 130]).■

Remark 1 Compared to [24], the graphon dynamical system model in (5) is time-varying; more specifically, the parameter matrices $A(\cdot), B(\cdot), D(\cdot)$ and $E(\cdot)$ are time-varying, but the underlying graphon $\mathbf{M}$ is time-invariant. This time-varying formulation will be used in characterizing the solutions to the

[1]The classical solution follows the definition of [42, Def. 2.1, Chp. 4], that is, $\mathbf{x}$ is continuous on $[0,T]$, $\mathbf{x}(t)$ is in the domain of $\mathbb{A}(t)$ for all $t \in [0,T]$, $\mathbf{x}$ is continuously differentiable on $(0,T)$ and satisfies (7) on $[0,T]$.

[2]That is the solution $\mathbf{x}$ is continuous on $[0,T]$, $\mathbf{x}(t)$ is in the domain of $[A(t)\mathbb{I} + D(t)\mathbf{M}]$ for all $t \in [0,T]$, $\mathbf{x}$ is continuously differentiable on $(0,T)$ and satisfies (5) on $[0,T]$.

limit LQG-GMFG problems via two coupled graphon time-varying differential equations (see Section IV-C). □

Remark 2 The graphon appearing twice in (5) can be replaced by graphons $\mathbf{M}_1$ in the first position and $\mathbf{M}_2$ in the second respectively. Results in Lemmas 1-2 still hold. For simplicity of presentation, we restrict the discussions only to the case where the two graphons are the same. □

B. Relations with Finite Network Systems

Consider an N-node network with the following nodal dynamics: for $i \in \{1, ..., N\}$,

$$\dot{x}_i(t) = A(t)x_i(t) + B(t)u_i(t) + D(t)x_i^{\mathcal{G}}(t) + E(t)u_i^{\mathcal{G}}(t) \quad (10)$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ represent respectively the state and the control of ith node at time t, and

$$x_i^{\mathcal{G}}(t) \triangleq \frac{1}{N}\sum_{j=1}^{N} m_{ij}x_j(t) \quad \text{and} \quad u_i^{\mathcal{G}}(t) \triangleq \frac{1}{N}\sum_{j=1}^{N} m_{ij}u_j(t)$$

represent respectively the network influence of states and that of the control at time $t \in [0, T]$. The coupling weights satisfy that $m_{ij} \leq c$ for all $i, j \in \{1, ..., N\}$ where c is the same constant for the graphon set $\mathcal{W}_c$. We note that problems with m-dimensional control inputs ($m < n$) for the nodal dynamics can be represented by placing zeros in columns (with indices between m and n) of $D(t)$ and $E(t)$ for all $t \in [0, T]$.

Consider a uniform partition $\{P_1, \ldots, P_N\}$ of $[0, 1]$ with $P_1 = [0, \frac{1}{N}]$ and $P_k = (\frac{k-1}{N}, \frac{k}{N}]$ for $2 \leq k \leq N$. The *step function graphon* $\mathbf{M}^{[\mathbf{N}]}$ that corresponds to $M_N \triangleq [m_{ij}]$ is defined by

$$\mathbf{M}^{[\mathbf{N}]}(\vartheta, \varphi) \triangleq \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbb{1}_{P_i}(\vartheta)\mathbb{1}_{P_j}(\varphi)m_{ij}, \quad (\vartheta, \varphi) \in [0, 1]^2,$$

where $\mathbb{1}_{P_i}(\cdot)$ is the indicator function (that is, $\mathbb{1}_{P_i}(\vartheta) = 1$ if $\vartheta \in P_i$ and $\mathbb{1}_{P_i}(\vartheta) = 0$ if $\vartheta \notin P_i$). Let $\mathbf{x}^{[\mathbf{N}]}(t) \in (L^2[0, 1])^n$ be the piece-wise constant function (in the ϑ argument) corresponding to $x(t) \triangleq (x_1(t)^\intercal, ..., x_N(t)^\intercal)^\intercal \in \mathbb{R}^{nN}$ given by $\mathbf{x}_\vartheta^{[\mathbf{N}]}(t) \triangleq \sum_{i=1}^{N} \mathbb{1}_{P_i}(\vartheta)x_i(t), \ \forall \vartheta \in [0, 1]$. Similarly, define $\mathbf{u}^{[\mathbf{N}]}(t) \in (L^2[0, 1])^n$ that corresponds to $u(t) \triangleq (u_1(t)^\intercal, ..., u_N(t)^\intercal)^\intercal \in \mathbb{R}^{nN}$.

Then the network system in (10) may be equivalently represented by the following graphon dynamical system

$$\begin{aligned}\dot{\mathbf{x}}^{[\mathbf{N}]}(t) &= \left[A(t)\mathbb{I} + D(t)\mathbf{M}^{[\mathbf{N}]}\right]\mathbf{x}^{[\mathbf{N}]}(t) \\ &+ \left[B(t)\mathbb{I} + D(t)\mathbf{M}^{[\mathbf{N}]}\right]\mathbf{u}^{[\mathbf{N}]}(t),\end{aligned} \quad (11)$$

where $t \in [0, T]$, $\mathbf{x}^{[\mathbf{N}]}(t), \mathbf{u}^{[\mathbf{N}]}(t) \in (L^2_{pwc[0,1]})^n$, $\mathbf{M}^{[\mathbf{N}]} \in \mathcal{W}_c$ represents step function graphon couplings associated with the underlying graph (via its adjacency matrix $[m_{ij}]$), and $L^2_{pwc}[0, 1]$ denotes the set of all piece-wise constant (over each element of the uniform partition) functions in $L^2[0, 1]$.

The trajectories of the graphon dynamical system in (11) correspond one-to-one to the trajectories of the network system in (10), following a similar proof argument to [21, Lem. 3]. Moreover, the system in (5) can represent the limit system for a sequence of systems represented in the form of (11) when the underlying step function graphon sequence converges to a limit graphon (under suitable norms) and initial conditions converges to a limit initial condition in $(L^2[0, 1])^n$, following a similar proof argument to [21, Thm. 7].

IV. LQG Graphon Mean Field Games

The application of the graphon mean field games methodology to finite network game problems is as follows: by passing to the nodal population limit and then network limit, one can identify the limit equilibrium; this limit equilibrium is then used by all the agents to generate the approximation of the best response strategies. This methodology bypasses the combinatorial intractability of computing the exact Nash equilibria for dynamic game problems on large networks.

A. Stochastic Dynamic Games on Finite Networks

Consider an N-node graph where each node is associated with a homogeneous population of individual agents. Each individual agent is influenced by the mean field of its nodal population and the mean fields of other nodal populations over the graph. Let $\mathcal{V}_c$ denote the set of nodes representing the clusters and $N = |\mathcal{V}_c|$ denote the total number of such nodes. Let $\mathcal{C}_q$ denote the set of agents in the qth cluster. Then the total number of agents is given by $K = \sum_{q=1}^{N} |\mathcal{C}_q|$.

Following the problem formulation in [2]–[4], the dynamics of an individual agent $i \in \{1, ..., K\}$ are given by

$$dx_i(t) = (Ax_i(t) + Bu_i(t) + Dz_i(t))dt + \Sigma dw_i(t), \quad (12)$$

where $t \in [0, T]$, $x_i(t)$, $u_i(t)$, and $z_i(t)$ are respectively the state, the control and the *network empirical average* in $\mathbb{R}^n$. $\{w_i, 1 \leq i \leq K\}$ are independent standard n-dimensional Wiener processes and are independent of the initial conditions $\{x_i(0), 1 \leq i \leq K\}$ which are also assumed to be independent. Σ is a constant $n \times n$ matrix. We drop the time index for $A(\cdot), B(\cdot), D(\cdot)$ purely for notation simplicity. Problems with m-dimensional control inputs ($m < n$) for the nodal dynamics can be represented by placing zeros in columns (with indices between m and n) of B. For an agent $i \in \mathcal{C}_q$, the *network empirical average* $z_i(t)$ is given by

$$z_i(t) = \frac{1}{N}\sum_{\ell=1}^{N} m_{q\ell}\frac{1}{|\mathcal{C}_\ell|}\sum_{j \in \mathcal{C}_\ell} x_j(t) = \frac{1}{N}\sum_{\ell=1}^{N} m_{q\ell}\bar{x}_\ell(t) \quad (13)$$

where $M = [m_{q\ell}]$ is the adjacency matrix of the underlying graph and $\bar{x}_\ell(t) \triangleq \frac{1}{|\mathcal{C}_\ell|}\sum_{j \in \mathcal{C}_\ell} x_j(t)$.

The individual agent's cost is given by

$$\begin{aligned}J_i(u_i) \triangleq \mathbb{E}\Big(\int_0^T \big(\|x_i(t) - \nu_i(t)\|_Q^2 + \|u_i(t)\|_R^2\big)dt \\ + \|x_i(T) - \nu_i(T)\|_{Q_T}^2\Big)\end{aligned} \quad (14)$$

where $Q, Q_T \geq 0, R > 0$, $\nu_i(t) \triangleq H(z_i(t) + \eta)$, $\eta \in \mathbb{R}^n$ and $H \in \mathbb{R}^{n \times n}$. In other words, agent i is trying to ensure that its state $x_i(t)$ tracks $\nu_i(t)$ for all $t \in [0, T]$ with relatively small control efforts. This cost mimics, for example, the phenomenon that in social networks individuals tend to conform with the population behavior (see e.g. [44]). The bias term when it is not zero provides the flexibility to model scenarios where individuals aim to achieve higher (or lower) states compared to the population (weighted) average state.

Let $\gamma_i(\cdot,\cdot):[0,T]\times\mathcal{I}_i\to\mathbb{R}^n$ denote the strategy of agent i, $i\in\{1,...,K\}$ where $\mathcal{I}_i$ denotes the information set available to agent i. The control of agent i at time t is then given by $u_i(t)=\gamma_i(t,y)$ with $y\in\mathcal{I}_i$. A strategy K-tuple $(\gamma_1,...,\gamma_K)$ is a *Nash equilibrium* if it satisfies

$$J_i(\gamma_i,\gamma_{-i})\le J_i(\gamma,\gamma_{-i}),\quad \forall\gamma(\cdot,\cdot):[0,T]\times\mathcal{I}_i\to\mathbb{R}^n,\tag{15}$$

for all $i\in\{1,...,K\}$, where $\gamma_{-i}\triangleq(\gamma_1,...,\gamma_{i-1},\gamma_{i+1},...,\gamma_K)$ and $J(\gamma,\gamma_{-i})$ denotes the cost for agent i when agent i follows strategy $\gamma(\cdot,\cdot):[0,T]\times\mathcal{I}_i\to\mathbb{R}^n$ and all the other agents follow strategies specified in γ_{-i}. Given that all other agents are taking strategies specified by γ_{-i}, the *best response* of agent i is defined by $\arg\inf_{\gamma\in\mathcal{U}_i}J_i(\gamma,\gamma_{-i})$, where the sets of admissible strategies $(\mathcal{U}_i)_{i=1}^K$ may consist of open-loop, close-loop, or state-feedback strategies depending on the information structures (see [30] for detailed discussions).

Directly finding Nash equilibria for such problems on very large networks is generally intractable. The graphon mean field game approach [2]–[4] employs the idea of finding approximate solutions based on both the mean field limit and the graphon limit. The corresponding best response for each individual agent in the approximate solution is decentralized in the sense that for each agent i only its local state observation is required in $\mathcal{I}_i$.

B. Infinite Nodal Population Problems on Finite Networks

In the asymptotic local population limit (i.e. $|\mathcal{C}_q|\to\infty$ for all $q\in\{1,...,N\}$), the dynamics of a generic agent α in the cluster $\mathcal{C}_q$ (i.e. $\alpha\in\mathcal{C}_q$) are then given by

$$dx_\alpha(t)=(Ax_\alpha(t)+Bu_\alpha(t)+Dz_\alpha(t))dt+\Sigma dw_\alpha(t),\tag{16}$$

where $z_\alpha(t)=\frac{1}{N}\sum_{\ell=1}^N m_{q\ell}\bar{x}_\ell(t)$ and

$$\bar{x}_\ell(t)\triangleq\lim_{|\mathcal{C}_\ell|\to\infty}\frac{1}{|\mathcal{C}_\ell|}\sum_{j\in\mathcal{C}_\ell}x_j(t).$$

The cost for a generic agent $\alpha\in\mathcal{C}_q$ is then

$$J_\alpha(u_\alpha)=\mathbb{E}\Big(\int_0^T\big(\|x_\alpha(t)-\nu_\alpha(t)\|_Q^2+\|u_\alpha(t)\|_R^2\big)dt + \|x_\alpha(T)-\nu_\alpha(T)\|_{Q_T}^2\Big),\tag{17}$$

where $Q,Q_T\ge 0,R>0$ and $\nu_\alpha(t)\triangleq H(z_\alpha(t)+\eta)$. Let

$$\bar{u}_\ell(t)\triangleq\lim_{|\mathcal{C}_\ell|\to\infty}\frac{1}{|\mathcal{C}_\ell|}\sum_{i\in\mathcal{C}_\ell}u_i(t),\;\; \bar{z}_\ell(t)\triangleq\lim_{|\mathcal{C}_\ell|\to\infty}\frac{1}{|\mathcal{C}_\ell|}\sum_{i\in\mathcal{C}_\ell}z_i(t).$$

Then clearly $\bar{z}_\ell(t)=z_\alpha(t)$ for all $\alpha\in\mathcal{C}_\ell$. Assuming the limits $\bar{u}_\ell,\bar{z}_\ell$ and $\bar{x}_\ell$ exist when $|\mathcal{C}_q|\to\infty$ for all $q\in\{1,...,N\}$, we obtain the dynamics in the infinite population limit for each cluster $\mathcal{C}_\ell$, $\ell\in\mathcal{V}_c$, from the individual agent dynamics given by (16); specifically, the dynamics of the *nodal mean field* $\bar{x}_\ell(t)$ in the cluster $\mathcal{C}_\ell$ satisfy the following

$$\dot{\bar{x}}_\ell(t)=A\bar{x}_\ell(t)+B\bar{u}_\ell(t)+D\bar{z}_\ell(t).\tag{18}$$

Then the *(nodal) network mean field* $\bar{z}_q(t)\triangleq\frac{1}{N}\sum_{\ell=1}^N m_{q\ell}\bar{x}_\ell(t)$ for node $q\in\mathcal{V}_c$ is given by the deterministic dynamics

$$\dot{\bar{z}}_q(t)=A\bar{z}_q(t)+\frac{1}{N}\sum_{\ell=1}^N m_{q\ell}(B\bar{u}_\ell(t)+D\bar{z}_\ell(t)),\quad q\in\mathcal{V}_c.$$

The *network mean field* refers to $\bar{z}(t)\triangleq(\bar{z}_1(t)^\intercal,...,\bar{z}_N(t)^\intercal)^\intercal$. Let $\bar{s}(t)$ and $\bar{x}(t)$ be defined similarly to $\bar{z}(t)$.

Proposition 1 *If there exists a unique (classical) solution pair $(\bar{z},\bar{s})$ to the coupled forward-backward equations*

$$\begin{aligned}\dot{\bar{z}}(t)=&\ \big(I_N\otimes A_c(t)+\frac{1}{N}M\otimes D\big)\bar{z}(t)\\ &-\frac{1}{N}M\otimes BR^{-1}B^\intercal\bar{s}(t),\ \bar{z}(0)=\frac{1}{N}(M\otimes I_n)\bar{x}(0),\end{aligned}\tag{19}$$

$$\begin{aligned}-\dot{\bar{s}}(t)=&\ I_N\otimes A_c(t)^\intercal\bar{s}(t)-I_N\otimes(QH-\Pi_tD)\bar{z}(t)\\ &-(I_N\otimes QH)(\mathbf{1}_N\otimes\eta),\\ \bar{s}(T)=&\ (I_N\otimes Q_TH)(\bar{z}(T)+\mathbf{1}_N\otimes\eta),\end{aligned}\tag{20}$$

where $t\in[0,T]$, $A_c(t)\triangleq(A-BR^{-1}B^\intercal\Pi_t)$, and $\Pi_{(\cdot)}$ is the solution to the $n\times n$-dimensional matrix Riccati equation

$$-\dot{\Pi}_t=A^\intercal\Pi_t+\Pi_tA-\Pi_tBR^{-1}B^\intercal\Pi_t+Q,\ \Pi_T=Q_T,\tag{21}$$

then the game problem defined by (16) *and* (17) *has a unique Nash equilibrium and the best response in the equilibrium is given as follows: for a generic agent α in cluster $\mathcal{C}_q$,*

$$u_\alpha(t)=-R^{-1}B^\intercal(\Pi_tx_\alpha(t)+\bar{s}_q(t)),\ \alpha\in\mathcal{C}_q,\ q\in\mathcal{V}_c.\tag{22}$$

PROOF Within an infinite nodal population, the individual effect on the nodal mean field is negligible. Hence each individual agent in cluster $\mathcal{C}_q$ is solving an LQG tracking problem to track a reference trajectory ν_q. The best response for a generic agent α in cluster $\mathcal{C}_q$ is simply given by the optimal LQG tracking solution in (22) where $\Pi_{(\cdot)}$ is given by (21) and $\bar{s}_q$ is given by

$$-\dot{\bar{s}}_q(t)=A_c(t)^\intercal\bar{s}_q(t)-Q\nu_q(t)+\Pi_tD\bar{z}_q(t),\tag{23}$$

with $\bar{s}_q(T)=Q_T\nu_q(T)$ and $\nu_q\triangleq H(\bar{z}_q+\eta)$. If all agents follow the best response in (22), then the evolution of the network mean field $\bar{z}$ must satisfy

$$\begin{aligned}\dot{\bar{z}}_q(t)=&\ A_c(t)\bar{z}_q(t)+D\frac{1}{N}\sum_{\ell=1}^N m_{q\ell}\bar{z}_\ell(t)\\ &-BR^{-1}B^\intercal\frac{1}{N}\sum_{\ell=1}^N m_{q\ell}\bar{s}_\ell(t)\end{aligned}\tag{24}$$

with $\bar{z}_q(0)=\frac{1}{N}\sum_{\ell=1}^N m_{q\ell}\bar{x}_\ell(0),\ 1\le q\le N$. If there exists a unique solution pair $(\bar{z}_q(t),\bar{s}_q(t))_{q\in\mathcal{V}_c,t\in[0,T]}$ to (24) and (23), then the best response strategy for each agent is uniquely determined by (22), (21), (23) and (24). The joint equations (23) and (24) can be represented in an equivalent compact form by two nN-dimensional equations as (19) and (20). ■

The solution pair to the two coupled equations (19) and (20) together with the sufficient conditions for existence and uniqueness can be provided based on the fixed-point method in [32] or the solution method based on Riccati equations following [45]–[47]. See [48] for more details on numerical procedures based on the two different methods.

Each individual agent, in order to generate the (network mean field) best response in (22), needs to solve two nN dimensional equations (19) and (20), and moreover each individual agent is required to know the exact graph structure. For large graphs, the computation of solutions and the requirement for exact graph structure become extremely difficult, if not intractable, to achieve. To overcome these difficulties, we employ the idea of approximating large graph structures by their graphon limit(s) in the following section.

C. Infinite Nodal Population Problems on Graphons

Consider a uniform partition $\{P_1, \ldots, P_N\}$ of $[0,1]$ with $P_1 = [0, \frac{1}{N}]$ and $P_k = (\frac{k-1}{N}, \frac{k}{N}]$ for $2 \le k \le N$. Let node q be associated with the partition P_q. If we embed $\bar{z}$ and $\bar{s}$ into the Hilbert space $L^2\big([0,T];(L^2[0,1])^n\big)$, denoted by $\mathbf{z}^{[\mathbf{N}]}$ and $\mathbf{s}^{[\mathbf{N}]}$ following the construction in Section III-B, then the problem given by (19) and (20) can be equivalently represented by the following graphon time-varying dynamical systems:

$$\begin{aligned}&\dot{\mathbf{z}}^{[\mathbf{N}]}(t) = \big(\mathbb{A}(t) + [D\mathbf{M}^{[\mathbf{N}]}]\big)\mathbf{z}^{[\mathbf{N}]}(t) - [BR^{-1}B^\intercal \mathbf{M}^{[\mathbf{N}]}]\mathbf{s}^{[\mathbf{N}]}(t)\\ &\mathbf{z}^{[\mathbf{N}]}(0) = \int_{[0,1]} \mathbf{M}(\cdot,\beta)\bar{x}_\beta(0)d\beta,\end{aligned} \tag{25}$$

$$\begin{aligned}\dot{\mathbf{s}}^{[\mathbf{N}]}(t) &= -\big[\mathbb{A}(t)^\intercal\big]\mathbf{s}^{[\mathbf{N}]}(t) + [(QH - \Pi_t D)\mathbb{I}]\mathbf{z}^{[\mathbf{N}]}(t)\\ &+ [QH\mathbb{I}](\eta\mathbf{1}), \quad \mathbf{s}^{[\mathbf{N}]}(T) = [Q_T H\mathbb{I}](\mathbf{z}^{[\mathbf{N}]}(T) + \eta\mathbf{1}),\end{aligned} \tag{26}$$

where $\mathbb{A}(t) \triangleq [(A - BR^{-1}B^\intercal \Pi_t)\mathbb{I}] \in \mathcal{L}((L^2[0,1])^n)$, and $\mathbf{z}^{[\mathbf{N}]}, \mathbf{s}^{[\mathbf{N}]} \in L^2([0,T];(L^2_{pwc}[0,1])^n)$.

This equivalent formulation enables us to represent arbitrary-size graphs, since any graph of a finite size can be represented by $\mathbf{M}^{[\mathbf{N}]}$ through a step function graphon as illustrated in Section III-B. As the number of nodes goes to infinity, the limit of joint equations (19) and (20) (if it exists) is given by the joint equations (27) and (28) below.

LQG-GMFG Forward-Backward Equations:

$$\begin{aligned}&\dot{\mathbf{z}}(t) = \big(\mathbb{A}(t) + [D\mathbf{M}]\big)\mathbf{z}(t) - [BR^{-1}B^\intercal \mathbf{M}]\mathbf{s}(t),\\ &\mathbf{z}(0) = [I\mathbf{M}]\bar{\mathbf{x}}(0) = \int_{[0,1]} \mathbf{M}(\cdot,\beta)\bar{x}_\beta(0)d\beta \in (L^2[0,1])^n,\end{aligned} \tag{27}$$

$$\begin{aligned}&\dot{\mathbf{s}}(t) = -\mathbb{A}(t)^\intercal \mathbf{s}(t) + [(QH - \Pi_t D)\mathbb{I}]\mathbf{z}(t) + [QH\mathbb{I}](\eta\mathbf{1}),\\ &\mathbf{s}(T) = [Q_T H\mathbb{I}](\mathbf{z}(T) + \eta\mathbf{1}) \in (L^2[0,1])^n,\end{aligned} \tag{28}$$

where $\mathbb{A}(t) \triangleq [(A - BR^{-1}B^\intercal \Pi_t)\mathbb{I}] \in \mathcal{L}((L^2[0,1])^n)$, $\Pi_{(\cdot)}$ is given by the $n \times n$-dimensional Riccati equation

$$-\dot{\Pi}_t = A^\intercal \Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\intercal \Pi_t + Q, \ \Pi_T = Q_T, \tag{29}$$

and $\mathbf{z}, \mathbf{s} \in L^2([0,T];(L^2[0,1])^n)$.

Conditions for the existence and uniqueness of solution pairs to the joint equations are presented later in Section V and Section VI, and the convergence properties of the solution pairs $\{(\mathbf{z}^{[\mathbf{N}]}, \mathbf{s}^{[\mathbf{N}]})\}$ to $(\mathbf{z}, \mathbf{s})$ in $C([0,T];(L^2[0,1])^n) \times C([0,T];(L^2[0,1])^n)$ under technical conditions are presented in [1], [48].

If the joint solutions $\mathbf{z}$ and $\mathbf{s}$ to (27) and (28) exist in $L^2([0,T];(L^2[0,1])^n)$, then by Lemma 1 they also lie in $C\big([0,T];(L^2[0,1])^n\big)$. By the Arzelà–Ascoli Theorem and the Uniform Limit Theorem [49], the space $C([0,T];(L^2[0,1])^n)$ is complete under the uniform norm $\|\cdot\|_C$ defined by

$$\|\mathbf{v}\|_C \triangleq \sup_{t\in[0,T]} \|\mathbf{v}(t)\|_2, \quad \forall \mathbf{v} \in C([0,T];(L^2[0,1])^n). \tag{30}$$

Proposition 2 *Assume there exists a unique classical solution pair* $(\mathbf{z}, \mathbf{s})$ *to equations* (27) *and* (28)*. Then the graphon limit mean field game problem has a unique Nash equilibrium and the best response in the equilibrium for a generic agent* α *in cluster* $\mathcal{C}_\vartheta$ *for almost all* $\vartheta \in [0,1]$ *is given by*

$$u_\alpha(t) = -R^{-1}B^\intercal(\Pi_t x_\alpha(t) + \mathbf{s}_\vartheta(t)), \ \alpha \in \mathcal{C}_\vartheta, \vartheta \in [0,1] \tag{31}$$

where $(\mathbf{s}_\vartheta(t))_{\vartheta\in[0,1],t\in[0,T]}$ *is given by the joint equations* (27) *and* (28)*, and* $\Pi_{(\cdot)}$ *is given by* (29). □

The proof follows the same lines of arguments as the proof for Proposition 1.

The best response in the Nash equilibrium for the limit LQG-GMFG problem is similar to that in [2]–[4], but the characterization of the offset process $\mathbf{s}$ is different. The LQG-GMFG Forward-Backward Equations explicitly specify the space for the solution pair $(\mathbf{z}, \mathbf{s})$ following similar lines to the analysis Graphon Control in [19], [21], [22], whereas in [2]–[4] these processes are specified in a pointwise sense. The formulation in this paper further enables the analysis of LQG-GMFG solutions based on spectral and subspace decompositions.

V. Solution via Fixed-Point Analysis

A. Sufficient Conditions for Existence and Uniqueness

Let $\mathbb{A}(t) \triangleq [(A - BR^{-1}B^\intercal \Pi_t)\mathbb{I}]$. Let $\phi_1^{\mathbf{M}}(\cdot,\cdot)$ and $\phi_2(\cdot,\cdot)$ denote the evolution operators respectively associated with $[\mathbb{A}(\cdot) + D\mathbf{M}]$ and $[-\mathbb{A}(\cdot)^\intercal]$. Following the standard definition of mild solutions in (6), the joint equations (27) and (28) have the following integral representations

$$\mathbf{z}(t) = \phi_1^{\mathbf{M}}(t,0)\mathbf{z}(0) + \int_0^t \phi_1^{\mathbf{M}}(t,\tau)(-[BR^{-1}B^\intercal \mathbf{M}]\mathbf{s}(\tau))d\tau, \tag{32}$$

$$\begin{aligned}\mathbf{s}(\tau) &= \phi_2(\tau,T)\mathbf{s}(T)\\ &- \int_\tau^T \phi_2(\tau,q)\big([(QH - \Pi_q D)\mathbb{I}]\mathbf{z}(q) + [QH\mathbb{I}]\eta\big)dq.\end{aligned} \tag{33}$$

Substituting $\mathbf{s}(\tau)$ in (32) by (33) yields

$$\begin{aligned}\mathbf{z}(t) &= \phi_1^{\mathbf{M}}(t,0)\mathbf{z}(0)\\ &- \int_0^t \phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal \mathbf{M}]\Big\{\phi_2(\tau,T)\mathbf{s}(T) -\\ &\int_\tau^T \phi_2(\tau,q)\big([(QH - \Pi_q D)\mathbb{I}]\mathbf{z}(q) + [QH\mathbb{I}]\eta\big)dq\Big\}d\tau.\end{aligned}$$

We recall from (28) that $\mathbf{s}(T) = [Q_T H\mathbb{I}](\mathbf{z}(T) + \eta\mathbf{1})$. Assuming the initial boundary condition $\mathbf{z}(0)$ is known, we then define the following operator $\Gamma(\cdot)$ from $L^2([0,T];(L^2[0,1])^n)$ to $L^2([0,T];(L^2[0,1])^n)$:

$$(\Gamma(\mathbf{v}))(t) \triangleq \phi_1^{\mathbf{M}}(t,0)\mathbf{z}(0) - \int_0^t \phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal\mathbf{M}]\Big\{\phi_2(\tau,T)[Q_TH\mathbb{I}](\mathbf{v}(T)+\eta\mathbf{1}) - \int_\tau^T \phi_2(\tau,q)\big([(QH-\Pi_qD)\mathbb{I}]\mathbf{v}(q)+[QH\mathbb{I}]\eta\mathbf{1}\big)dq\Big\}d\tau \tag{34}$$

for any $\mathbf{v}$ in $L^2([0,T];(L^2[0,1])^n)$. Then one can easily verify the following lemma.

Lemma 3 *$\Gamma(\cdot)$ is a mapping from $C([0,T];(L^2[0,1])^n)$ to $C([0,T];(L^2[0,1])^n)$.* □

Lemma 3 allows us to use the Contraction Mapping Principle in the Banach space $C\big([0,T];(L^2[0,1])^n\big)$ endowed with the uniform norm in (30) to establish conditions for the existence of a unique solution pair to the joint equations (27) and (28). Define the $\mathbf{M}$-dependent constant

$$\mathrm{L}_0(\mathbf{M}) \triangleq \sup_{t\in[0,T]}\Big\{\int_0^t\int_\tau^T \Big\|\Big\{\phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal\mathbf{M}]\phi_2(\tau,q)[(QH-\Pi_qD)\mathbb{I}]\Big\}\Big\|_{\text{op}} dq d\tau\Big\} + \sup_{t\in[0,T]}\Big\{\int_0^t \Big\|\phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal\mathbf{M}]\phi_2(\tau,T)[Q_TH\mathbb{I}]\Big\|_{\text{op}} d\tau\Big\}.$$

Lemma 4 *If the following condition*

$$\mathrm{L}_0(\mathbf{M}) < 1 \tag{35}$$

holds, then there exists a unique classical solution pair $(\mathbf{z},\mathbf{s})$ to the joint equations (27) and (28).

The proof follows the standard contraction argument (see e.g. [50]). See Appendix A-A for the proof.

Several factors could decrease the amplitude of $\mathrm{L}_0(\mathbf{M})$, which include (i) a small value of T and (ii) weak couplings via small norms of Q, Q_T and D.

B. Spectral Decompositions of Forward-Backward Equations

Let $\mathcal{S}_*$ denote the characterizing invariant subspace of $\mathbf{M}$ as defined in Section II-B and let $\mathcal{S}_*^\perp$ denote the orthogonal complement of $\mathcal{S}_*$ in $L^2[0,1]$. Consider all the orthonormal eigenfunctions $\{\mathbf{f}_\ell\}_{\ell\in\mathcal{I}_\lambda}$ of $\mathbf{M}$ associated with eigenvalues $\{\lambda_\ell\}_{\ell\in\mathcal{I}_\lambda}$, where $\mathcal{I}_\lambda$ denotes the index multiset for all the non-zero eigenvalues of $\mathbf{M}$. By the definition of the characterizing invariant subspace, we have $\mathcal{S}_* = \text{span}(\mathbf{f}_\ell, \ell\in\mathcal{I}_\lambda)$. Since the graphon operator $\mathbf{M}$ defined in (1) is a Hilbert–Schmidt integral operator and hence a compact operator in $\mathcal{L}(L^2[0,1])$, the number of elements in $\mathcal{I}_\lambda$ is either finite or countably infinite (see for instance [17, Prop. 1]). Projecting the processes $\mathbf{z}$ and $\mathbf{s}$ governed by (27) and (28) into the orthogonal subspaces $(\mathcal{S}_*^\perp)^n$ and the eigendirections $(\text{span}(\mathbf{f}_\ell))^n$ with $\ell\in\mathcal{I}_\lambda$ yields the following result.

Proposition 3 *If the joint equations (27) and (28) have a unique classical solution pair $(\mathbf{z},\mathbf{s})$, then the solution pair satisfies the following: for almost all $\theta\in[0,1]$ and for all $t\in[0,T]$,*

$$\begin{aligned}\mathbf{s}_\theta(t) &= \sum_{\ell\in\mathcal{I}_\lambda}\mathbf{f}_\ell(\theta)s^\ell(t) + \breve{s}(t)(1-\sum_{\ell\in\mathcal{I}_\lambda}\langle\mathbf{f}_\ell,\mathbf{1}\rangle\mathbf{f}_\ell(\theta)))\\ &= \sum_\ell \mathbf{f}_\ell(\theta)(s^\ell(t)-\breve{s}(t)) + \breve{s}(t),\\ \mathbf{z}_\theta(t) &= \sum_{\ell\in\mathcal{I}_\lambda}\mathbf{f}_\ell(\theta)z^\ell(t),\end{aligned} \tag{36}$$

where for all $\ell\in\mathcal{I}_\lambda$, $z^\ell(t)\mathbf{f}_\ell \in (\text{span}(\mathbf{f}_\ell))^n$ and $s^\ell(t)\mathbf{f}_\ell \in (\text{span}(\mathbf{f}_\ell))^n$, $\breve{s}(t)\big(\mathbf{1}-\sum_{\ell\in\mathcal{I}_\lambda}\langle\mathbf{f}_\ell,\mathbf{1}\rangle\mathbf{f}_\ell\big)\in(\mathcal{S}_*^\perp)^n$, and z^ℓ, s^ℓ, $\breve{s}\in C([0,T];\mathbb{R}^n)$ are given by

$$\begin{aligned}\dot{z}^\ell(t) &= (A_c(t)+\lambda_\ell D)z^\ell(t) - \lambda_\ell BR^{-1}B^\intercal s^\ell(t),\\ z^\ell(0) &= \lambda_\ell\int_{[0,1]}\mathbf{f}_\ell(\beta)\bar{x}_\beta(0)d\beta,\end{aligned} \tag{37}$$

$$\begin{aligned}\dot{s}^\ell(t) &= -A_c(t)^\intercal s^\ell(t) + (QH-\Pi_tD)z^\ell(t) + QH\eta,\\ s^\ell(T) &= Q_TH(z^\ell(T)+\eta),\end{aligned} \tag{38}$$

$$\dot{\breve{s}}(t) = -A_c(t)^\intercal\breve{s}(t) + QH\eta, \quad \breve{s}(T) = Q_TH\eta, \tag{39}$$

with $A_c(t)\triangleq(A-BR^{-1}B^\intercal\Pi_t)$.

Proof For $t\in[0,T]$, let the component-wise decomposition of $\mathbf{s}(t)$ be given as follows: $\mathbf{s}(t) = \sum_{\ell\in\mathcal{I}_\lambda}\mathbf{s}^{\mathbf{f}_\ell}(t) + \breve{\mathbf{s}}(t)$, $\mathbf{s}^{\mathbf{f}_\ell}(t)\in(\text{span}(\mathbf{f}_\ell))^n, \breve{\mathbf{s}}(t)\in(\mathcal{S}_*^\perp)^n$, where $\mathcal{S}_*^\perp$ denotes the orthogonal complement subspace of $\mathcal{S}_*\triangleq\text{span}(\{\mathbf{f}_\ell\}_{\ell\in\mathcal{I}_\lambda})$ in $L^2[0,1]$ and $\mathcal{I}_\lambda$ denotes the index multiset for all the non-zero eigenvalues of $\mathbf{M}$. Similarly define $\mathbf{z}^{\mathbf{f}_\ell}(t)$ and $\breve{\mathbf{z}}(t)$. The following operators $\mathbb{A}(t)^\intercal$, $[(QH-\Pi_tD)\mathbb{I}]$, $[QH\mathbb{I}]$, $[\mathbb{A}(t)+D\mathbf{M}]$ and $[BR^{-1}B^\intercal\mathbf{M}]$, $[I\mathbf{M}]$ and $[Q_TH\mathbb{I}]$ share the same invariant subspaces $(\mathbf{f}_\ell)^n$, $(\mathcal{S}_*^\perp)^n$ and $\mathcal{S}_*^n$ (see [24, Prop. 3]). Hence the dynamics (27) and (28) can be component-wise decoupled into different subspaces (similar to that in [24, Lem. 2]). Furthermore if we let $\mathbf{z}^{\mathbf{f}_\ell}(t) = z^\ell(t)\mathbf{f}_\ell$ and $\mathbf{s}^{\mathbf{f}_\ell}(t) = s^\ell(t)\mathbf{f}_\ell$, then for any matrix $F\in\mathbb{R}^{n\times n}$,

$$\begin{aligned}&[F\mathbf{M}]\mathbf{z}^{\mathbf{f}_\ell}(t) = \lambda_\ell[F\mathbb{I}]\mathbf{z}^{\mathbf{f}_\ell}(t) = \lambda_\ell Fz^\ell(t)\mathbf{f}_\ell \in (L^2[0,1])^n,\\ &[F\mathbf{M}]\mathbf{s}^{\mathbf{f}_\ell}(t) = \lambda_\ell[F\mathbb{I}]\mathbf{s}^{\mathbf{f}_\ell}(t) = \lambda_\ell Fs^\ell(t)\mathbf{f}_\ell \in (L^2[0,1])^n,\\ &[F\mathbf{M}]\breve{\mathbf{z}}(t) = [F\mathbf{M}]\breve{\mathbf{s}}(t) = 0 \in (L^2[0,1])^n, \quad \forall t\in[0,t].\end{aligned}$$

We note that $\breve{\mathbf{z}}(t) = \mathbf{z}(t) - \sum_{\ell\in\mathcal{I}_\lambda}\mathbf{z}^{\mathbf{f}_\ell}(t) = \mathbf{z}(t) - \sum_{\ell\in\mathcal{I}_\lambda}z^\ell(t)\mathbf{f}_\ell$ and similar representations hold for $\breve{\mathbf{s}}$. We note that since $\mathbf{s}$ and $\mathbf{z}$ are classical solutions, $\dot{\mathbf{s}}(\cdot)$ and $\dot{\mathbf{z}}(\cdot)$ are well defined in $C([0,T];(L^2[0,1])^n)$. Hence the projections of $\dot{\mathbf{s}}(t)$ and $\dot{\mathbf{z}}(t)$ into eigen subspaces are well-defined for $t\in[0,T]$. By projecting both sides of (27) and (28) into different eigendirections $\{(\mathbf{f}_\ell)^n\}_{\ell\in\mathcal{I}_\lambda}$ and the orthogonal subspace $(S_*^\perp)^n$, we obtain the following:

$$\begin{aligned}\dot{\mathbf{s}}^{\mathbf{f}_\ell}(t) &= -\big[\mathbb{A}(t)^\intercal\big]\mathbf{s}^{\mathbf{f}_\ell}(t) + [(QH-\Pi_tD)\mathbb{I}]\mathbf{z}^{\mathbf{f}_\ell}(t)\\ &\qquad + [QH\mathbb{I}](\langle\mathbf{f}_\ell,\mathbf{1}\rangle\eta\mathbf{f}_\ell)\\ \mathbf{s}^{\mathbf{f}_\ell}(T) &= [Q_TH\mathbb{I}](\mathbf{z}^{\mathbf{f}_\ell}(T)+\langle\mathbf{f}_\ell,\mathbf{1}\rangle\eta\mathbf{f}_\ell)\in(L^2[0,1])^n,\end{aligned} \tag{40}$$

$$\begin{aligned}\dot{\mathbf{z}}^{\mathbf{f}_\ell}(t) &= [\mathbb{A}(t)+\lambda_\ell D\mathbb{I}]\mathbf{z}^{\mathbf{f}_\ell}(t) - [\lambda_\ell BR^{-1}B^\intercal\mathbb{I}]\mathbf{s}^{\mathbf{f}_\ell}(t)\\ \mathbf{z}^{\mathbf{f}_\ell}(0) &= \Big(\lambda_\ell\int_{[0,1]}\mathbf{f}_\ell(\beta)\bar{x}_\beta(0)d\beta\Big)\mathbf{f}_\ell,\end{aligned} \tag{41}$$

$$\dot{\breve{\mathbf{z}}}(t) = [\mathbb{A}(t)]\breve{\mathbf{z}}(t), \qquad \mathbf{z}(0) = 0 \in (\mathcal{S}_*^{\perp})^n, \tag{42}$$

(which implies $\breve{\mathbf{z}}(t) = 0$ for all $t \in [0,T]$, and hence)

$$\begin{aligned} \dot{\breve{\mathbf{s}}}(t) &= -\big[\mathbb{A}(t)^{\intercal}\big]\breve{\mathbf{s}}(t) + [QH\mathbb{I}](\eta(\mathbf{1} - \sum_{\ell\in\mathcal{I}_\lambda}\langle \mathbf{f}_\ell, \mathbf{1}\rangle \mathbf{f}_\ell)), \\ \breve{\mathbf{s}}(T) &= [Q_T H\mathbb{I}](\eta(\mathbf{1} - \sum_{\ell\in\mathcal{I}_\lambda}\langle \mathbf{f}_\ell, \mathbf{1}\rangle \mathbf{f}_\ell)) \in (\mathcal{S}_*^{\perp})^n. \end{aligned} \tag{43}$$

Let $\breve{\mathbf{s}}(t) = \breve{s}(t)(\mathbf{1} - \sum_{\ell\in\mathcal{I}_\lambda}\langle \mathbf{f}_\ell, \mathbf{1}\rangle \mathbf{f}_\ell) \in (\mathcal{S}_*^{\perp})^n$. Equivalently, we have (36), (37), (38) and (39). We note that $\breve{\mathbf{z}}(t)$ for $t \in [0,T]$ is always zero. ■

Remark 3 (Solution Complexity) It is worth emphasizing that (37), (38) and (39) are all n-dimensional differential equations. (39) always has a solution, but the joint equations (37) and (38) require extra conditions for the existence of a unique solution pair. The solution pair to the joint equations (37) and (38) can be numerically computed via fixed-point iterations (see Algorithm 1 in [48]). Let d_{dist} denote the number of distinct non-zero eigenvalues of $\mathbf{M}$. Then each agent only needs to solve one n-dimensional differential equation as (39) and d_{dist} number of forward-backward joint equation pairs as (37) and (38), each of which is n-dimensional. We note that $d_{\text{dist}} \leq \text{rank}(\mathbf{M})$. If d_{dist} is infinite, one may rely on approximations via a finite number of eigendirections. A special case of the equations (37) and (38) is studied in [27].□

VI. Solutions via an Operator Riccati Equation

A. Sufficient Conditions for Existence and Uniqueness

Following the standard idea for decoupling finite dimensional coupled forward-backward differential equations in [45]–[47] (which is a well-established technique called the invariant imbedding [51]), we decouple the infinite dimensional coupled forward-backward equations (27) and (28) based on the following non-symmetric operator Riccati equation

$$\begin{aligned} -\dot{\mathbb{P}} =&\mathbb{A}(t)^{\intercal}\mathbb{P} + \mathbb{P}\mathbb{A}(t) + \mathbb{P}[D\mathbf{M}] - \mathbb{P}[BR^{-1}B^{\intercal}\mathbf{M}]\mathbb{P} \\ &-[(QH - \Pi_t D)\mathbb{I}], \quad \mathbb{P}(T) = [Q_T H\mathbb{I}] \end{aligned} \tag{44}$$

where $\mathbb{A}(t) \triangleq (A - BR^{-1}B^{\intercal}\Pi_t)\mathbb{I} \in \mathcal{L}((L^2[0,1])^n)$ and $\Pi_{(\cdot)}$ solves the $n\times n$-dimensional Riccati equation in (29). Same as infinite dimensional Riccati equations in [43], the operator Riccati equation above is defined through its solutions.

Let $I \in \mathbb{R}$ denote a compact time interval. A mapping $\mathbb{F} : I \to \mathcal{L}((L^2[0,1])^n)$ is *strongly continuous* if $\mathbb{F}(\cdot)\mathbf{v}$ is continuous for all $\mathbf{v}$ in $(L^2[0,1])^n$. A sequence $\{\mathbb{F}_n\}$ of strongly continuous mappings *converges strongly* to $\mathbb{F}$ if for all $\mathbf{v} \in (L^2[0,1])^n$, the following holds: $\lim_{n\to\infty}\sup_{t\in I}\|\mathbb{F}_n(t)\mathbf{v} - \mathbb{F}(t)\mathbf{v}\|_2 = 0$. Let $C_s(I;\mathcal{L}((L^2[0,1])^n)$ denote the set of strongly continuous mappings I to $\mathcal{L}((L^2[0,1])^n$ under the strong convergence defined above. The strong continuity of $\mathbb{P} \in C_s(I;\mathcal{L}((L^2[0,1])^n)$ implies that for each $\mathbf{v} \in (L^2[0,1])^n$, $\mathbb{P}(\cdot)\mathbf{v}$ is bounded over the compact time interval I. Hence by the Uniform Boundedness Principle, $\|\mathbb{P}(\cdot)\|_{\text{op}}$ is uniformly bounded, that is, $\sup_{t\in I}\|\mathbb{P}(t)\|_{\text{op}} < \infty$ (see [43]). Let $C_u([0,T];\mathcal{L}((L^2[0,1])^n)$ denote the space of strongly continuous mappings endowed with uniform norm $\|\mathbb{F}\| := \sup_{t\in I}\|\mathbb{F}(t)\|_{\text{op}}$. We note that for any compact interval $I \subset \mathbb{R}$, the spaces $C_u(I;\mathcal{L}((L^2[0,1])^n)$ and $C_s(I;\mathcal{L}((L^2[0,1])^n)$ are equal as sets but their topologies are different (see [43]).

Definition 2 (Mild Solution to Operator Riccati Eqn.) $\mathbb{P} \in C_s([0,T];\mathcal{L}((L^2[0,1])^n)$ is a mild solution to (44) if it satisfies the following equation for all $\mathbf{v} \in (L^2[0,1])^n$,

$$\begin{aligned} \mathbb{P}(t)\mathbf{v} = \mathbb{P}(T)\mathbf{v} + \int_t^T &\Big(\mathbb{A}(\tau)^{\intercal}\mathbb{P}(\tau) + \mathbb{P}(\tau)(\mathbb{A}(\tau) + [D\mathbf{M}]) \\ &-\mathbb{P}(\tau)[BR^{-1}B^{\intercal}\mathbf{M}]\mathbb{P}(\tau) - [(QH - \Pi_\tau D)\mathbb{I}]\Big)\mathbf{v}d\tau \end{aligned} \tag{45}$$

with terminal condition $P(T) = [Q_T H\mathbb{I}]$. □

Proposition 4 (Strong Differentiability of Mild Solutions) *If the mild solution $\mathbb{P}$ in $C_s([0,T];\mathcal{L}((L^2[0,1])^n)$ to the operator Riccati equation* (44) *exists, then it is strongly differentiable (that is, for any $\mathbf{v} \in (L^2[0,1])^n$, $\mathbb{P}(\cdot)\mathbf{v}$ is differentiable).*

Proof If the mild solution $\mathbb{P}$ in $C_s([0,T];\mathcal{L}((L^2[0,1])^n)$ to the operator Riccati equation (44) exists, then one can verify that the integrand on the right-hand side of (45) is continuous. Hence for any $\mathbf{v}$, $\mathbb{P}(\cdot)\mathbf{v}$ is differentiable. ■

Lemma 5 (Product Rule) *Let $\mathbb{P}$ be the mild solution to* (44) *and $(\mathbf{z},\mathbf{s})$ be the classical solution pair to* (27) *and* (28)*. Then following product rule holds*

$$\frac{d(\mathbb{P}(t)\mathbf{z}(t))}{dt} = \lim_{\varepsilon\to 0}\frac{(\mathbb{P}(t+\varepsilon) - \mathbb{P}(t))\mathbf{z}(t)}{\varepsilon} + \mathbb{P}(t)\dot{\mathbf{z}}(t). \tag{46}$$

See Appendix A-B for the proof.

Consider the following assumption

- **(A1)** The operator Riccati equation (44) has a unique mild solution $\mathbb{P} \in C_s([0,T];\mathcal{L}((L^2[0,1])^n)$.

Lemma 6 *Let $\mathbb{A}_1 \in C_s([0,T];\mathcal{L}((L^2[0,1])^n)$ and $\mathbf{u} \in C([0,T];(L^2[0,1])^n)$. Then the following system*

$$\dot{\mathbf{x}}(t) = \mathbb{A}_1(t)\mathbf{x}(t) + \mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_o \in (L^2[0,1])^n, \tag{47}$$

has a unique classical solution $\mathbf{x}$.

See Appendix A-C for the proof.

Proposition 5 (Classical Solution Pair) *If* (A1) *holds, then the joint equations* (27) *and* (28) *have a unique classical solution pair $(\mathbf{z},\mathbf{s})$ with $\mathbf{z},\mathbf{s}$ in $C([0,T];(L^2[0,1])^n)$.* □

Proof First let us assume that the classical solution pair $(\mathbf{z},\mathbf{s})$ to (27) and (28) exists. Let $\mathbf{e}(t) \triangleq \mathbf{s}(t) - \mathbb{P}(t)\mathbf{z}(t)$ for all $t \in [0,T]$. Then we can apply the product rule in Lemma 5 and obtain that

$$\begin{aligned} \dot{\mathbf{e}}(t) &= \dot{\mathbf{s}}(t) - \lim_{\varepsilon\to 0}\frac{(\mathbb{P}(t+\varepsilon) - \mathbb{P}(t))\mathbf{z}(t)}{\varepsilon} - \mathbb{P}(t)\dot{\mathbf{z}}(t) \\ &= -\mathbb{A}(t)^{\intercal}\mathbf{s}(t) + [QH\mathbb{I}](\mathbf{z}(t) + \eta\mathbf{1}) - [\Pi_t D\mathbb{I}]\mathbf{z}(t) \\ &\quad + \mathbb{A}(t)^{\intercal}\mathbb{P}(t)\mathbf{z}(t) + \mathbb{P}(t)\mathbb{A}(t)\mathbf{z}(t) + \mathbb{P}(t)[D\mathbf{M}]\mathbf{z}(t) \\ &\quad - \mathbb{P}(t)[BR^{-1}B^{\intercal}\mathbf{M}]\mathbb{P}(t)\mathbf{z}(t) - [(QH - \Pi_t D)\mathbb{I}]\mathbf{z}(t) \\ &\quad - \mathbb{P}(t)\left(\mathbb{A}(t)\mathbf{z}(t) + [D\mathbf{M}]\mathbf{z}(t) - [BR^{-1}B^{\intercal}\mathbf{M}]\mathbf{s}(t)\right) \\ &= \left(-\mathbb{A}(t)^{\intercal} + \mathbb{P}(t)[BR^{-1}B^{\intercal}\mathbf{M}]\right)\mathbf{e}(t) + [QH\mathbb{I}](\eta\mathbf{1}). \end{aligned} \tag{48}$$

That is $\mathbf{e}$ is the classical solution to

$$\dot{\mathbf{e}}(t) = \left(-\mathbb{A}(t)^\intercal + \mathbb{P}(t)[BR^{-1}B^\intercal\mathbf{M}]\right)\mathbf{e}(t) + [QH\mathbb{I}](\eta\mathbf{1}) \tag{49}$$

with $\mathbf{e}(T) = \mathbf{s}(T) - \mathbb{P}(T)\mathbf{z}(T) = [Q_T H\mathbb{I}](\eta\mathbf{1})$. Substituting $\mathbf{s}(t)$ in (27) by $\mathbf{s}(t) = \mathbb{P}(t)\mathbf{z}(t) + \mathbf{e}(t)$ yields

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \big(\mathbb{A}(t) + [D\mathbf{M}] - [BR^{-1}B^\intercal\mathbf{M}]\mathbb{P}(t)\big)\mathbf{z}(t)\\ &\quad -[BR^{-1}B^\intercal\mathbf{M}]\mathbf{e}(t),\\ \mathbf{z}(0) &= \int_{[0,1]}\mathbf{M}(\cdot,\beta)\bar{x}_\beta(0)d\beta \in (L^2[0,1])^n,\end{aligned} \tag{50}$$

Now without assuming the classical solution pair $(\mathbf{z},\mathbf{s})$ exists, under Assumption (A1), one first computes $\mathbf{e}$ following equation (48). Then based on $\mathbf{e}$ and $\mathbb{P}$, one computes $\mathbf{z}$ following (50). Finally, one computes $\mathbf{s}$ based on equation (28) and $\mathbf{z}$. Based on Lemma 6, when Assumption (A1) holds, each of these equations (48), (50) and (28) has a unique classical solution. Furthermore, one can verify that the pair $(\mathbf{z},\mathbf{s})$ generated following this procedure is actually the classical solution pair to the joint forward backward equation (27) and (28). Therefore we obtain the unique classical solution pair $(\mathbf{z},\mathbf{s})$ for (27) and (28). ∎

Remark 4 In the proof, the joint equations (27) and (28) are decoupled based on the solution of the operator Riccati solution (44). Moreover, given the solution to (44), the proof actually provides a direct procedure for computing the solution pair to the joint equations (27) and (28) by introducing a new process $\mathbf{e} \in C([0,T];(L^2[0,1])^n)$ in (48) that satisfies $\mathbf{e}(t) = \mathbf{s}(t) - \mathbb{P}\mathbf{z}(t), t \in [0,T]$. □

B. Subspace Decomposition for Operator Riccati Equations

Consider any subspace $\mathcal{S} \in L^2[0,1]$ and its orthogonal complement $\mathcal{S}^\perp$ in $L^2[0,1]$. An operator $\bar{\mathbb{T}} \in \mathcal{L}((L^2[0,1])^n)$ is called the $(\mathcal{S})^n$*-equivalent operator* of $\mathbb{T} \in \mathcal{L}((L^2[0,1])^n)$ if the following holds

$$\bar{\mathbb{T}}\mathbf{v} = \mathbb{T}\mathbf{v} \quad \text{and} \quad \bar{\mathbb{T}}\mathbf{u} = 0, \quad \forall \mathbf{v} \in (\mathcal{S})^n,\ \forall \mathbf{u} \in (\mathcal{S}^\perp)^n. \tag{51}$$

Let $\bar{\mathbb{P}}(t) \in \mathcal{L}((L^2[0,1])^n)$ denote the $(\mathcal{S})^n$-equivalent operator of $\mathbb{P}(t)$. Let $\mathbb{I}_\mathcal{S}$ (resp. $\mathbb{I}_{\mathcal{S}^\perp}$) in $\mathcal{L}(L^2[0,1])$ denote the $\mathcal{S}$-equivalent operator (resp. $\mathcal{S}^\perp$-equivalent operator) of the identity operator $\mathbb{I} \in \mathcal{L}(L^2[0,1])$. Let $A_c(t) \triangleq (A - BR^{-1}B^\intercal\Pi_t)$. Let the subspace $\mathcal{S}_* \subset L^2[0,1]$ be the characterizing graphon invariant subspace of $\mathbf{M}$ as defined in Section II-B and let $\mathcal{S}_*^\perp$ denote its orthogonal complement subspace in $L^2[0,1]$.

Theorem 1 (Riccati Equation Subspace Decomposition) *If* (A1) *holds, then the solution to the non-symmetric operator Riccati equation* (44) *is given by*

$$\mathbb{P}(t) = [P^\perp(t)\mathbb{I}_{\mathcal{S}_*^\perp}] + \bar{\mathbb{P}}(t), \quad t \in [0,T] \tag{52}$$

where $[P^\perp(t)\mathbb{I}_{\mathcal{S}_*^\perp}] \in \mathcal{L}((\mathcal{S}_*^\perp)^n)$, $\bar{\mathbb{P}}(t) \in \mathcal{L}((\mathcal{S}_*)^n)$ *is given by the non-symmetric operator Riccati equation*

$$\begin{aligned} -\dot{\bar{\mathbb{P}}}(t) =&\ [A_c(t)\mathbb{I}_{\mathcal{S}_*}]^\intercal\bar{\mathbb{P}}(t) + \bar{\mathbb{P}}(t)[A_c(t)\mathbb{I}_{\mathcal{S}_*}] + \bar{\mathbb{P}}(t)[D\mathbf{M}]\\ &-\bar{\mathbb{P}}(t)[BR^{-1}B^\intercal\mathbf{M}]\bar{\mathbb{P}}(t) - [(QH - \Pi_t D)\mathbb{I}_{\mathcal{S}_*}],\\ \mathbb{P}(T) =&\ [Q_T H\mathbb{I}_{\mathcal{S}_*}], \quad t \in [0,T],\end{aligned} \tag{53}$$

and $P^\perp(t) \in \mathbb{R}^{n\times n}$ *is given by the* $n \times n$*-dimensional linear matrix differential equation*

$$\begin{aligned}-\dot{P}^\perp(t) &= A_c(t)^\intercal P^\perp(t) + P^\perp(t)A_c(t) - (QH - \Pi_t D),\\ P^\perp(T) &= \gamma Q_T,\ t \in [0,T].\end{aligned} \tag{54}$$

Proof Let $[\Theta(t)] : (L^2[0,1])^n \to (L^2[0,1])^n$ with $t \in [0,T]$ denote the operator that corresponds to the right-hand side of the operator Riccati equation (44), that is, for $t \in [0,T]$

$$\begin{aligned}[\Theta(\mathbb{P}(t))] \triangleq&\ \mathbb{A}(t)^\intercal\mathbb{P}(t) + \mathbb{P}(t)\mathbb{A}(t) + \mathbb{P}(t)[D\mathbf{M}]\\ &-\mathbb{P}(t)[BR^{-1}B^\intercal\mathbf{M}]\mathbb{P} - [(QH - \Pi_t D)\mathbb{I}].\end{aligned}$$

Let $\mathcal{S}$ be any invariant subspace of $\mathbf{M}$. Then both $(\mathcal{S})^n$ and $(\mathcal{S}^\perp)^n$ are invariant subspaces of $[\Theta(\mathbb{P}(t))]$. For any $\mathbf{v} \in (L^2[0,1])^n$, there exists a unique component-wise decomposition $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}^\perp$ where $\bar{\mathbf{v}} \in (\mathcal{S})^n$ and $\mathbf{v}^\perp \in (\mathcal{S}^\perp)^n$ (see Section II-B). Then

$$\begin{aligned}[\Theta(\mathbb{P}(t))]\mathbf{v} &= [\Theta(\mathbb{P}(t))]\bar{\mathbf{v}} + [\Theta(\mathbb{P}(t))]\mathbf{v}^\perp\\ &= [\Theta(\bar{\mathbb{P}}(t))]\bar{\mathbf{v}} + [\Theta(\mathbb{P}^\perp(t))]\mathbf{v}^\perp\end{aligned} \tag{55}$$

where $\bar{\mathbb{P}}(t)$ (resp. $\mathbb{P}^\perp(t)$) is the $(\mathcal{S})^n$-equivalent operator (resp. $(\mathcal{S}^\perp)^n$ equivalent operator) of $\mathbb{P}(t)$. Assumption (A1) enforces the existence of a unique mild solution to the Riccati equation (44). By Prop. 4, the solution is strongly differentiable, and hence we obtain

$$\begin{aligned}-\frac{d}{dt}\{(\bar{\mathbb{P}}(t) + \mathbb{P}^\perp(t))\mathbf{v}\} &= -\frac{d}{dt}\{(\bar{\mathbb{P}}(t)\bar{\mathbf{v}}\} - \frac{d}{dt}\{\mathbb{P}^\perp(t))\mathbf{v}^\perp\}\\ &= [\Theta(\bar{\mathbb{P}}(t))]\bar{\mathbf{v}} + [\Theta(\mathbb{P}^\perp(t))]\mathbf{v}^\perp.\end{aligned} \tag{56}$$

Therefore, based on the property (51) of equivalent operators, we obtain the following decoupled equations:

$$\begin{aligned}-\frac{d}{dt}\{(\bar{\mathbb{P}}(t)\bar{\mathbf{v}}\} &= [\Theta(\bar{\mathbb{P}}(t))]\bar{\mathbf{v}},\\ -\frac{d}{dt}\{\mathbb{P}^\perp(t))\mathbf{v}^\perp\} &= [\Theta(\mathbb{P}^\perp(t))]\mathbf{v}^\perp\end{aligned} \tag{57}$$

with terminal conditions $\bar{\mathbb{P}}(T) = [\gamma Q_T\mathbb{I}]\bar{\mathbf{v}}$ and $\mathbb{P}^\perp(T) = [Q_T H\mathbb{I}]\mathbf{v}^\perp$. If we specialize $\mathcal{S}$ to the characterizing invariant subspace $\mathcal{S}_*$ of $\mathbf{M}$, then we obtain explicitly

$$\begin{aligned}-\frac{d}{dt}\{(\bar{\mathbb{P}}(t)\bar{\mathbf{v}}\} &= [A_c(t)\mathbb{I}]^\intercal\bar{\mathbb{P}}\bar{\mathbf{v}} + \bar{\mathbb{P}}[A_c(t)\mathbb{I}]\bar{\mathbf{v}} + \bar{\mathbb{P}}[D\mathbf{M}]\bar{\mathbf{v}}\\ &\quad - \bar{\mathbb{P}}[BR^{-1}B^\intercal\mathbf{M}]\bar{\mathbb{P}}\bar{\mathbf{v}} - [(QH - \Pi_t D)\mathbb{I}]\bar{\mathbf{v}},\\ -\frac{d}{dt}\{(\mathbb{P}^\perp(t)\mathbf{v}^\perp\} &= [A_c(t)\mathbb{I}]^\intercal\mathbb{P}^\perp\mathbf{v}^\perp + \mathbb{P}^\perp[A_c(t)\mathbb{I}]\mathbf{v}^\perp\\ &\quad - [(QH - \Pi_t D)\mathbb{I}]\bar{\mathbf{v}}^\perp\end{aligned}$$

with terminal conditions $\bar{\mathbb{P}}(T) = [\gamma Q_T\mathbb{I}]\bar{\mathbf{v}}$ and $\mathbb{P}^\perp(T) = [Q_T H\mathbb{I}]\mathbf{v}^\perp$. Since the choice of $\mathbf{v} \in (L^2[0,1])^n$ is arbitrary, the solution for $\bar{\mathbb{P}}(\cdot)$ is equivalently given by the strongly differentiable solution to (53) and the solution for $\mathbb{P}^\perp(\cdot)$ is given by $\mathbb{P}^\perp(\cdot) = [P^\perp(\cdot)\mathbb{I}_{\mathcal{S}_*^\perp}]$ with $P^\perp(\cdot)$ satisfying (54). Therefore the solution to the Riccati equation (44) is given by (52), (53) and (54). ∎

Remark 5 The key property that allows the decomposition in (57) in the proof of Theorem 1 is that the parameter operators $[A_c(t)\mathbb{I}]$, $[D\mathbf{M}]$, $[BR^{-1}B^\intercal\mathbf{M}]$, $[(QH - \Pi_t D)\mathbb{I}]$ and

$[Q_T H\mathbb{I}]$ in the Riccati equation (44) share the same orthogonal invariant subspaces $(\mathcal{S})^n$ and $(\mathcal{S}^\perp)^n$ (see [24, Prop. 3]). Hence such decompositions can be generalized to Riccati equations with general parameter operators in $\mathcal{L}((L^2[0,1])^n)$ where the parameter operators are only required to share some common orthogonal invariant subspaces $(\mathcal{S})^n$ and $(\mathcal{S}^\perp)^n$. □

Let $\{\mathbf{f}_\ell\}_{\ell\in\mathcal{I}_\lambda}$ be the orthonormal eigenfunctions of $\mathbf{M}$ where $\mathcal{I}_\lambda$ denotes the index multiset for all the non-zero eigenvalues of $\mathbf{M}$. Let λ_ℓ be the eigenvalue of $\mathbf{M}$ corresponding to $\mathbf{f}_\ell$.

Proposition 6 (Riccati Equation Spectral Decomposition) *If Assumption* (A1) *holds, then the solution to the operator Riccati equation* (44) *is equivalently given by*

$$\begin{aligned}\mathbb{P}(t) &= [P^\perp(t)\mathbb{I}_{\mathcal{S}_*^\perp}] + \sum_{\ell\in\mathcal{I}_\lambda}[\bar{P}^\ell(t)\mathbf{f}_\ell\mathbf{f}_\ell^\intercal] \\ &= [P^\perp(t)\mathbb{I}] + \sum_{\ell\in\mathcal{I}_\lambda}[(\bar{P}^\ell(t)-P^\perp(t))\mathbf{f}_\ell\mathbf{f}_\ell^\intercal],\end{aligned} \tag{58}$$

where $t\in[0,T]$, $\mathbb{P}(t)\in\mathcal{L}((L^2[0,1])^n)$, $[P^\perp(t)\mathbb{I}]\in\mathcal{L}((L^2[0,1])^n)$, $[(\bar{P}^\ell(t)-P^\perp(t))\mathbf{f}_\ell\mathbf{f}_\ell^\intercal]\in\mathcal{L}((\mathcal{S}_*)^n)$, $P^\perp(t)\in\mathbb{R}^{n\times n}$ *is given by the* $n\times n$*-dimensional matrix differential equation* (54)*, and* $\bar{P}^\ell(t)\in\mathbb{R}^{n\times n}$, $\ell\in\mathcal{I}_\lambda$*, is given by the following non-symmetric matrix Riccati equation*

$$\begin{aligned}&-\dot{\bar{P}}^\ell(t) = A_c(t)^\intercal\bar{P}^\ell(t)+\bar{P}^\ell(t)(A_c(t)+\lambda_\ell D)\\ &-\lambda_\ell(t)\bar{P}^\ell BR^{-1}B^\intercal\bar{P}^\ell(t)-(QH-\Pi_t D),\quad \bar{P}^\ell(T)=Q_TH.\end{aligned} \tag{59}$$

□

The proof follows similar arguments as that for Theorem 1 by iteratively applying the decompositions into eigenspaces.

Remark 6 Each agent only needs to solve d_{dist} number of $n\times n$-dimensional Riccati equations as (59) and one $n\times n$-dimensional matrix differential equation as (54), where d_{dist} denotes the number of distinct non-zero eigenvalues of $\mathbf{M}$. We note that $d_{\text{dist}}\le\text{rank}(\mathbf{M})$. If d_{dist} is infinite, one may rely on approximations via a finite number of eigendirections. □

Remark 7 The decomposition of $\mathbb{P}$ in Prop. 6 and the dynamics of $\mathbf{e}$ and $\mathbf{z}$ in (48) and (50) allow us to project the processes $\mathbf{e}$ and $\mathbf{z}$ into different eigen directions (similar to those projections in Prop. 3); furthermore, the relation $\mathbf{s}(t)=\mathbb{P}(t)\mathbf{z}(t)+\mathbf{e}(t)$ for all $t\in[0,T]$ allows the projections of $\mathbf{s}$ into different eigen directions as well. □

C. Finite-Rank Graphon Case

Consider the following finite-rank assumption:

(A2) The characterizing graphon invariant subspace $\mathcal{S}$ of the limit graphon $\mathbf{M}\in\mathcal{W}_c$ is finite dimensional with dimension d.

Under Assumption (A2), let $\{\mathbf{f}_1,...,\mathbf{f}_d\}$ be the orthonormal basis functions for the characterizing graphon invariant subspace $\mathcal{S}_*$ in $L^2[0,1]$ (which are not necessarily eigenfunctions of $\mathbf{M}$). For a matrix $Q=[q_{\ell h}]\in\mathbb{R}^{nd\times nd}$ with $q_{\ell h}\in\mathbb{R}^{n\times n}$ for $\ell,h\in\{1,...,d\}$, let $[Q\mathbf{f}\mathbf{f}^\intercal]\triangleq\sum_{\ell=1}^d\sum_{h=1}^d[q_{\ell h}\mathbf{f}_\ell\mathbf{f}_h^\intercal]$, that is, for almost all $(x,y)\in[0,1]^2$, $[Q\mathbf{f}\mathbf{f}^\intercal](x,y)\triangleq\sum_{\ell=1}^d\sum_{h=1}^d q_{\ell h}\mathbf{f}_\ell(x)\mathbf{f}_h(y)$. Let the elements of $M_\mathbf{f}\in\mathbb{R}^{d\times d}$ be given by $M_{\mathbf{f}\ell h}=\langle\mathbf{f}_\ell,\mathbf{M}\mathbf{f}_h\rangle$, for all $\ell,h\in\{1,\dots,d\}$.

Applying Theorem 1 and identifying the coefficients under the orthonormal basis for $\mathcal{S}_*$, we obtain the following result.

Proposition 7 (Finite-Rank Spectral Decomposition) *Assume* (A1) *and* (A2) *hold. Then the solution to the non-symmetric operator Riccati equation* (44) *is given by*

$$\begin{aligned}\mathbb{P}(t) &= [P^\perp(t)\mathbb{I}_{\mathcal{S}_*^\perp}]+[\bar{P}(t)\mathbf{f}\mathbf{f}^\intercal]\\ &= [P^\perp(t)\mathbb{I}]+[(\bar{P}(t)-I_d\otimes P^\perp(t))\mathbf{f}\mathbf{f}^\intercal],\quad t\in[0,T],\end{aligned}$$

where $[P^\perp(t)\mathbb{I}_{\mathcal{S}_*^\perp}]\in\mathcal{L}((\mathcal{S}_*^\perp)^n)$, $[\bar{P}(t)\mathbf{f}\mathbf{f}^\intercal]\in\mathcal{L}(\mathcal{S}_*^n)$, $[(\bar{P}(t)-I_d\otimes P^\perp(t))\mathbf{f}\mathbf{f}^\intercal]\in\mathcal{L}(\mathcal{S}_*^n)$, $[P^\perp(t)\mathbb{I}]\in\mathcal{L}((L^2[0,1])^n)$, $P^\perp(t)\in\mathbb{R}^{n\times n}$ *is given by* (54)*, and* $\bar{P}(t)\in\mathbb{R}^{nd\times nd}$ *is given by the* $dn\times dn$*-dimensional non-symmetric Riccati equation*

$$\begin{aligned}-\dot{\bar{P}}(t) &= (I_d\otimes A_c(t)^\intercal)\bar{P}(t)+\bar{P}(t)(I_d\otimes A_c(t)+M_\mathbf{f}\otimes D)\\ &\quad -\bar{P}(t)(M_\mathbf{f}\otimes BR^{-1}B^\intercal)\bar{P}(t)-I_d\otimes(QH-\Pi_tD),\\ \bar{P}(T) &= I_d\otimes Q_TH,\quad t\in[0,T].\end{aligned}$$

□

VII. Relations between Two Solution Approaches

Proposition 8 (Relation between Sufficient Conditions) *If* $\mathrm{L}_0(\mathbf{M})<1$*, then* (A1) *holds (that is, the operator Riccati equation* (44) *has a unique mild solution).* □

See Appendix B for the proof.

This implies the assumption (A1) in the decoupling operator Riccati equation is not more restrictive than the contraction condition in the fixed-point approach.

We present the following result to illustrate the connection between the spectral decomposition of the forward-backward equations in Prop. 3 and the spectral decomposition of the operator Riccati equation in Prop. 6.

Proposition 9 *Let* $\Pi_{(\cdot)}$ *denote the solution to* (29) *and let* $A_c(t)\triangleq(A-BR^{-1}B^\intercal\Pi_t)$*. If the following* $n\times n$*-dimensional non-symmetric Riccati equation*

$$\begin{aligned}-\dot{o}_t^\ell &= A_c(t)^\intercal o_t^\ell+o_t^\ell(A_c(t)+\lambda_\ell D)-\lambda_\ell o_t^\ell BR^{-1}B^\intercal o_t^\ell\\ &\quad -(QH-\Pi_tD),\quad o^\ell(T)=Q_TH,\end{aligned} \tag{60}$$

has a unique solution, then the joint equations (37) *and* (38) *have a unique solution pair* (z^ℓ,s^ℓ) *in* $C([0,T];\mathbb{R}^n)$ *and furthermore* s^ℓ *and* z^ℓ, $\ell\in\mathcal{I}_\lambda$*, are respectively given by*

$$\dot{z}^\ell(t)=(A_c(t)+\lambda_\ell D)z^\ell(t)-\lambda_\ell BR^{-1}B^\intercal(e^\ell(t)+o_t^\ell z^\ell(t)), \tag{61}$$

$$s^\ell(t)=o_t^\ell z^\ell(t)+e^\ell(t), \tag{62}$$

with the initial condition $z^\ell(0)=\lambda_\ell\int_{[0,1]}\mathbf{f}_\ell(\beta)\bar{x}_\beta(0)d\beta$*, where* $e^\ell(t)$ *is given by*

$$\dot{e}^\ell(t)=\big(-A_c^\intercal(t)+\lambda_\ell o_t^\ell BR^{-1}B^\intercal\big)e^\ell(t)+QH\eta \tag{63}$$

with terminal condition $e^\ell(T)=Q_TH\eta$.

PROOF The proof follows the standard procedure for decoupling finite dimensional joint forward-backward equations (see e.g. [45]–[47]). Let $e^\ell(t) = s^\ell(t) - o^\ell_t z^\ell(t)$. Then

$$\begin{aligned}\dot{e}^\ell(t) =&\dot{s}^\ell(t) - \dot{o}^\ell_t z^\ell(t) - o^\ell_t \dot{z}^\ell(t)\\ =& - A_c^\mathsf{T}(t)s^\ell(t) + (QH - \Pi_t D)z^\ell(t) + QH\eta\\ &+ \Big(A_c(t)^\mathsf{T} o^\ell_t + o^\ell_t(A_c(t)+\lambda_\ell D) - \lambda_\ell o^\ell_t BR^{-1}B^\mathsf{T} o^\ell_t\\ &\quad - (QH - \Pi_t D)\Big) z^\ell(t)\\ &- \dot{o}^\ell_t\Big((A_c(t)+\lambda_\ell D)z^\ell(t) - \lambda_\ell BR^{-1}B^\mathsf{T} s^\ell(t)\Big)\\ =&\big(- A_c^\mathsf{T}(t) + \lambda_\ell o^\ell_t BR^{-1}B^\mathsf{T}\big)e^\ell(t) + QH\eta\end{aligned} \tag{64}$$

with terminal condition $e^\ell(T) = s^\ell(T) - o^\ell_T z^\ell(T) = Q_T H\eta$.

If (60) has a unique solution, then the solution e^ℓ to (64) always exists. Based on e^ℓ and o^ℓ, we can then obtain z^ℓ following (61). Finally, we can obtain s^ℓ based on (62). Clearly, z^ℓ and s^ℓ generated from the procedure above always satisfy equations (37) and (38). ■

Remark 8 We note that the Riccati equation in (60) is the same as that in (59) in Prop. 6. □

VIII. EXAMPLES

A. Example 1: Uniform Attachment (UA) Graphs

1) Uniform Attachment Procedure and the Graphon Limit: Uniform attachment graphs are generated as follows: (S1) Start with stage $k = 2$ and repeat the following steps (S2)-(S3); (S2) Add an edge with probability $\frac{1}{k}$ to each node pair that is not connected; (S3) Increase the stage number k by 1.

The sequence of random graphs generated based on the uniform attachment procedure converges to the limit graphon $\mathbf{M}(x, y) = 1 - \max(x, y),\ x, y \in [0, 1]$, under the cut metric with probability 1 (see [8, Prop. 11.40]).

Proposition 10 (Spectral Decomposition of UA Graphon) *All the eigen pairs for the uniform attachment graphon limit* $\mathbf{M}(x, y) = 1 - \max(x, y), x, y \in [0, 1]$ *are given by*

$$\left(\sqrt{2}\cos\left(\frac{k\pi(\cdot)}{2}\right), \frac{4}{k^2\pi^2}\right), \qquad k \in \{1, 3, 5, ..\}, \tag{65}$$

that is, the spectral decomposition of $\mathbf{M}$ *is given by*

$$\mathbf{M}(x, y) = \sum_{k=1,3,...} \frac{4}{k^2\pi^2}\sqrt{2}\cos\left(\frac{\pi k x}{2}\right)\sqrt{2}\cos\left(\frac{\pi k y}{2}\right), \tag{66}$$

with $x, y \in [0, 1]$. □

See Appendix C for the proof.

Since $\mathbf{M}$ is continuous and all its eigenvalues are non-negative, an application of Mercer's Theorem [52, Thm.3.1, Ch.4.] implies that the series in (66) converges uniformly and absolutely.

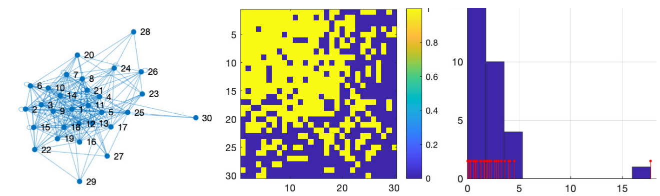

Fig. 2: A random graph (left) generated following the uniform attachment procedure, its pixel representation (middle) and the distribution of modulus of eigenvalues (right).

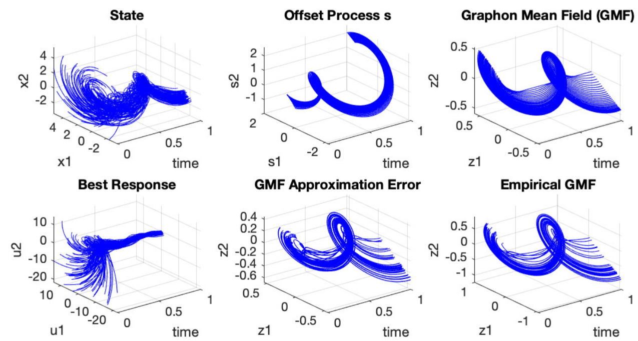

Fig. 3: Simulations on the uniform attachment graph example in Fig. 2 with 30 nodes where each node contains 4 agents and each agent has 2 states.

2) Simulations on Uniform Attachment Graphs: The parameters in the simulations are:

$$\begin{aligned}&A = \begin{bmatrix}0 & 10\\ -10 & 0\end{bmatrix},\ Q = \begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix},\ \Sigma = \begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix},\\ &B = D = R = Q_T = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \eta = \begin{bmatrix}2\\ 2\end{bmatrix},\ H = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\\ &\ T = 1, n = 2,\ N = 30,\ |C_\ell| = 4,\ 1 \le \ell \le N.\end{aligned} \tag{67}$$

The graphon limit is approximated by the 5 most significant eigen directions. We observe that the approximation by the 5 most significant eigen directions of $\mathbf{M}$ has less than 1%

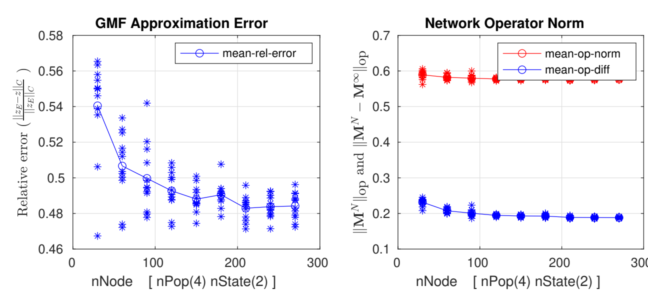

Fig. 4: The relative error in the graphon mean field decreases as graph sizes increase. The horizontal axis represents the number of nodes on the network denoted by nNode. 12 independent simulation experiments are carried out for each size. The nodal population size denoted by nPop is 4, the local state dimension denoted by nState is 2. In the figure on the right, blue dots represent the values for $\|\mathbf{M}^{[\mathbf{N}]} - \mathbf{M}\|_{\text{op}}$ in different simulation experiments.

relative error in terms of the operator norm. The initial conditions are independently generated from Gaussian distributions with variance 1 and means that are sampled from a uniform distribution in $[-3,3]$. These means are used in computing the approximate graphon mean field game solutions. In the example in Fig. 2 and Fig. 3, the graphon mean field (GMF) approximation relative error $\frac{\|\mathbf{z}_E-\mathbf{z}\|_C}{\|\mathbf{z}_E\|_C}$ is 52.569% where $\mathbf{z}_E$ is the $L^2_{pwc}[0,1]$ function associated with the network empirical average and $\mathbf{z}$ is the graphon mean field computed based on the LQG-GMFG Forward-Backward Equations. The error between the graphon limit $\mathbf{M}$ and the step function graphon $\mathbf{M}^{[\mathbf{N}]}$ (associated with the 30-node graph in Fig. 2) is $\|\mathbf{M}-\mathbf{M}^{[\mathbf{N}]}\|_{\text{op}} = 0.238$ and the graphon limit operator norm is $\|\mathbf{M}\|_{\text{op}} = 0.386$. The relative approximation errors decrease as the graphs increase in size, which is numerically illustrated by a set of examples for graphs with different sizes in Fig. 4.

B. Example 2: Stochastic Block Models (SBM)

1) Random Graphs Generated from SBM and Properties: Following [8, p.157], random simple graphs with N nodes can be generated from a graphon $\mathbf{M}$ by first sampling data points $x_1, ..., x_N$ from the uniform distribution on $[0,1]$ and then connecting node i and node j with probability $\mathbf{M}(x_i, x_j)$, for all $i, j \in \{1, ..., N\}$. Stochastic block models can be approximately considered as graphon models for generating random graphs where the underlying graphon $\mathbf{M}$ is a step function graphon (see [53]). Consider the stochastic block model matrix $W = [w_{ij}] \in \mathbb{R}^{d\times d}$. The associated graphon limit is given by $\mathbf{M}(x,y) = \sum_{i=1}^d \sum_{j=1}^d w_{ij} \mathbb{1}_{P_i}(x)\mathbb{1}_{P_j}(y),\ (x,y) \in [0,1]^2$ with the uniform partition $\{P_1, ..., P_d\}$ of $[0,1]$. Denote the eigen decomposition $W = \sum_{\ell=1}^d \lambda_\ell v_\ell v_\ell^\intercal$ where $\{\lambda_\ell\}$ are the eigenvalues (allowing repeated eigenvalues) and $\{v_\ell\}$ are the associated normalized eigenvectors. Then the spectral decomposition of the associated graphon is given by $\mathbf{M}(x,y) = \sum_{\ell=1}^d \frac{\lambda_\ell}{d} \mathbf{v}_\ell(x)\mathbf{v}_\ell(y),\ (x,y) \in [0,1]^2$ where $\mathbf{v}^\ell(x) = \sum_{i=1}^d \mathbb{1}_{P_i}(x) v_\ell(i)$ (see also [17]). The step function graphon is a low-rank graphon with the same number of nonzero eigenvalues as that of the block matrix W.

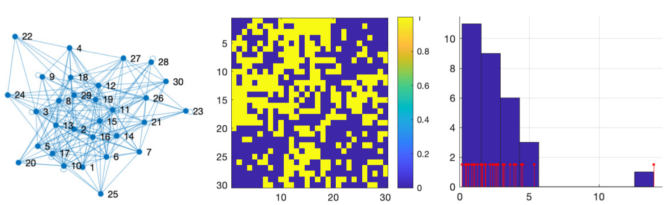

Fig. 5: A graph (left) generated from SBM, its pixel diagram (middle) and the distribution of modulus of eigenvalues (right).

2) Simulations on Random Graphs Generated from SBM: The parameters in the simulation are the same as those in (67). The initial conditions are independently drawn from Gaussian distributions with variance 1 and mean values that are generated randomly from $[-3,3]$. These mean values are used in computing the approximate graphon mean field game solutions. The block matrix of SBM is given by

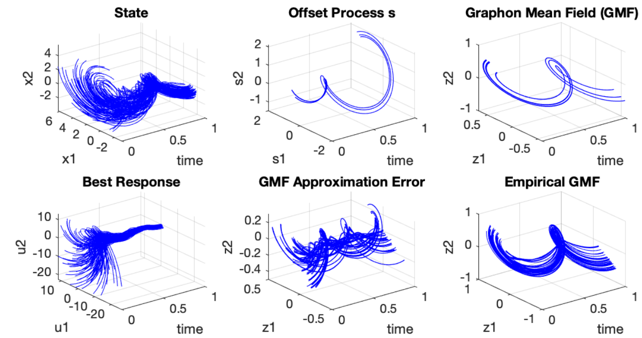

Fig. 6: Simulation on a network generated from SBM with 30 nodes where each node contains 4 agents and each agent has 2 states.

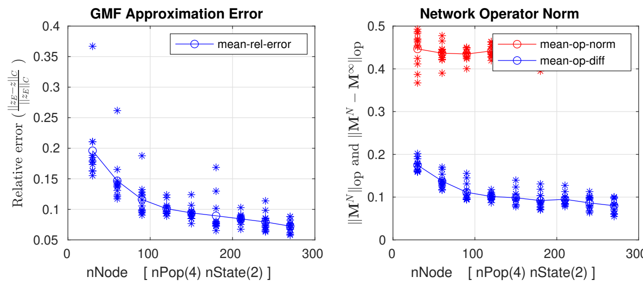

Fig. 7: Graphon mean field game approximation errors on networks of different sizes. The horizontal axis represents the number of nodes on the network denoted by nNode. 12 simulations are carried out for each size. The nodal population size denoted by nPop is 4, and the local state dimension denoted by nState is 2. In the figure on the right, blue dots represent values for $\|\mathbf{M}^{[\mathbf{N}]} - \mathbf{M}\|_{\text{op}}$ in different simulation experiments.

$$W = \begin{bmatrix} 0.25 & 0.5 & 0.2 \\ 0.5 & 0.35 & 0.7 \\ 0.2 & 0.7 & 0.4 \end{bmatrix}. \tag{68}$$

The simulation result on the graph instance in Fig. 5 which is generated from the SBM with matrix (68) is illustrated in Fig. 6. For this particular example, the graphon mean field relative approximation error $\frac{\|\mathbf{z}_E-\mathbf{z}\|_C}{\|\mathbf{z}_E\|_C}$ is 29.256% where $\mathbf{z}_E$ is the $L^2_{pwc}[0,1]$ function associated with the network empirical average and $\mathbf{z}$ is the graphon mean field computed based on the LQG-GMFG Forward-Backward Equations. The error between the graphon limit $\mathbf{M}$ and the step function graphon $\mathbf{M}^{[\mathbf{N}]}$ (associated with the graph) is $\|\mathbf{M}-\mathbf{M}^{[\mathbf{N}]}\|_{\text{op}} = 0.178$ and the graphon limit operator norm is $\|\mathbf{M}\| = 0.434$. The relative approximation error $\frac{\|\mathbf{z}_E-\mathbf{z}\|_C}{\|\mathbf{z}_E\|_C}$ decreases as the size of the network increases as illustrated by results on graphs of different sizes in Fig. 7.

IX. Conclusion

This work studied solution methods for LQG graphon mean field game problems based on subspace and spectral decompositions. Future work should focus on cases with heterogeneous parameters in dynamics, computational procedures for nonlinear graphon mean field games, graphon control for nonlinear systems, and the counterpart theory for sparse graphs.

Appendix A
Proof of Lemma 4, Lemma 5 and Lemma 6

A. Proof of Lemma 4

Proof For any $\mathbf{v},\mathbf{u}\in C([0,T];(L^2[0,1])^n)$,

$$\begin{aligned}
&\|\Gamma(\mathbf{v})-\Gamma(\mathbf{u})\|_C\\
&\le \sup_{t\in[0,T]}\Big\|\int_0^t \phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal\mathbf{M}]\phi_2(\tau,T)[Q_T H\mathbb{I}]\\
&\qquad\qquad (\mathbf{u}(T)-\mathbf{v}(T))d\tau\Big\|_2\\
&\quad+\sup_{t\in[0,T]}\Big\|\int_0^t\int_\tau^T\Big\{\phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal\mathbf{M}]\phi_2(\tau,q)\\
&\qquad\qquad [(QH-\Pi_q D)\mathbb{I}](\mathbf{v}(q)-\mathbf{u}(q))\Big\}dqd\tau\Big\|_2\\
&\le\Big\{\sup_{t\in[0,T]}\int_0^t\Big\|\phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal\mathbf{M}]\phi_2(\tau,T)[\gamma Q_T\mathbb{I}]\Big\|_{\text{op}}d\tau\\
&\quad+\sup_{t\in[0,T]}\int_0^t\int_\tau^T\Big\|\Big\{\phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal\mathbf{M}]\phi_2(\tau,q)\\
&\qquad\qquad [(QH-\Pi_q D)\mathbb{I}]\Big\}\Big\|_{\text{op}}dqd\tau\Big\}\,\|(\mathbf{v}-\mathbf{u})\|_C\\
&=\mathrm{L}_0(\mathbf{M})\|(\mathbf{v}-\mathbf{u})\|_C.
\end{aligned}\tag{69}$$

Therefore (35) ensures that $\Gamma(\cdot)$ is a contraction in the Banach space $C([0,T];(L^2[0,1])^n)$ endowed with the uniform norm $\|\cdot\|_C$ defined in (30). By the Contraction Mapping Principle, there exists a unique fixed point $\mathbf{z}\in C([0,T];(L^2[0,1])^n)$ for $\Gamma(\cdot)$. Based on (28), a unique solution $\mathbf{s}\in C([0,T];(L^2[0,1])^n)$ can then be obtained. Therefore LQG-GMFG Forward Backward Equations (27) and (28) have a unique mild solution pair $(\mathbf{z},\mathbf{s})$. Applying Lemma 2 to each of the equations (27) and (28), we obtain that $\mathbf{z}$ and $\mathbf{s}$ are also classical solutions. ■

B. Proof of Lemma 5

Proof To simplify the notation in the proof, we use $\mathbb{P}_t$ (resp. $\mathbf{z}_t$) to denote $\mathbb{P}(t)$ (resp. $\mathbf{z}(t)$). Based on the definition of (classical) differentiation, we have

$$\begin{aligned}
&\frac{d(\mathbb{P}(t)\mathbf{z}(t))}{dt}:=\lim_{\varepsilon\to0}\frac{\mathbb{P}_{t+\varepsilon}\mathbf{z}_{t+\varepsilon}-\mathbb{P}_t\mathbf{z}_t}{\varepsilon}\\
&=\lim_{\varepsilon\to0}\frac{(\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t)\mathbf{z}_{t+\varepsilon}}{\varepsilon}+\lim_{\varepsilon\to0}\frac{\mathbb{P}_t(\mathbf{z}_{t+\varepsilon}-\mathbf{z}_t)}{\varepsilon}\\
&=\lim_{\varepsilon\to0}\frac{(\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t)\mathbf{z}_t}{\varepsilon}+\mathbb{P}_t\dot{\mathbf{z}}_t+\lim_{\varepsilon\to0}\frac{(\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t)(\mathbf{z}_{t+\varepsilon}-\mathbf{z}_t)}{\varepsilon}.
\end{aligned}$$

Hence to prove the product rule, it is enough to show $\lim_{\varepsilon\to0}\frac{(\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t)(\mathbf{z}_{t+\varepsilon}-\mathbf{z}_t)}{\varepsilon}=0$, for all $t\in[0,T]$. Based on equation (27) and the fact that $\mathbf{z}$ and $\mathbf{s}$ are continuous in $C([0,T];(L^2[0,1])^n)$, we know

$$\begin{aligned}
\mathbf{z}_{t+\varepsilon}-\mathbf{z}_t&=\int_t^{t+\varepsilon}\big([\mathbb{A}(\tau)+DM]\mathbf{z}_\tau-[BR^{-1}B^\intercal\mathbf{M}]\mathbf{s}_\tau\big)\,d\tau\\
&=\big([\mathbb{A}(t)+DM]\mathbf{z}_t-[BR^{-1}B^\intercal\mathbf{M}]\mathbf{s}_t\big)\,\varepsilon+o(\varepsilon).
\end{aligned}$$

The second equality is due to the fact that the integrand is continuous and hence it is uniformly continuous over $[0,T]$ (see e.g. [54, Thm. 4.19, p. 91]). Then

$$\begin{aligned}
&\frac{(\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t)(\mathbf{z}_{t+\varepsilon}-\mathbf{z}_t)}{\varepsilon}\\
&=(\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t)\big([\mathbb{A}(t)+DM]\mathbf{z}_t-[BR^{-1}B^\intercal\mathbf{M}]\mathbf{s}_t\big)\\
&\quad+(\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t)W_\varepsilon
\end{aligned}$$

where W_ε denotes the element in $(L^2[0,1])^n$ with norm amplitude $o(1)$. We recall that the strong continuity of $\mathbb{P}\in C_s([0,T];\mathcal{L}((L^2[0,1])^n)$ means that $\mathbb{P}(\cdot)\mathbf{w}$ is continuous for any $\mathbf{w}\in(L^2[0,1])^n$. Hence, for any fixed $t\in[0,T]$,

$$\big\|(\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t)\big([\mathbb{A}(t)+DM]\mathbf{z}_t-[BR^{-1}B^\intercal\mathbf{M}]\mathbf{s}_t\big)\big\|_2$$

goes to zero as $\varepsilon\to0$. In addition, $\|(\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t)W_\varepsilon\|_2\le\|\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t\|_{\text{op}}\|W_\varepsilon\|_2$ goes to zero as $\varepsilon\to0$, since $\|\mathbb{P}_{t+\varepsilon}-\mathbb{P}_t\|_{\text{op}}$ is bounded as a result of the strong continuity of $\mathbb{P}$ and $\|W_\varepsilon\|_2\to0$ as $\varepsilon\to0$. Therefore we obtain (46). ■

C. Proof of Lemma 6

Proof By the definition of strong continuity of $\mathbb{P}\in C_s([0,T];\mathcal{L}((L^2[0,1])^n)$ and the Uniform Boundedness Principle, we obtain that $\|\mathbb{P}(\cdot)\|_{\text{op}}$ is uniformly bounded over the time interval $[0,T]$, that is, $\alpha:=\sup_{t\in[0,T]}\|\mathbb{P}(t)\|_{\text{op}}<\infty$. Define a mapping $\mathbb{T}$ from $C([0,T];(L^2[0,1])^n)$ to itself by $\mathbb{T}(\mathbf{x})(t)=\mathbf{x}_o+\int_0^t\Big(\mathbb{A}_1(\tau)\mathbf{x}(\tau)+\mathbf{u}(\tau)\Big)d\tau$. Recall that $\|\mathbf{x}\|_C:=\sup_{t\in[0,T]}\|\mathbf{x}(t)\|_2$. It easy to verify that $\|\mathbb{T}(\mathbf{x})(t)-\mathbb{T}(\mathbf{y})(t)\|_2\le t\alpha\|\mathbf{x}-\mathbf{y}\|_C$. Then by induction $\|\mathbb{T}^n(\mathbf{x})(t)-\mathbb{T}^n(\mathbf{y})(t)\|_2\le\frac{\alpha^n t^n}{n!}\|\mathbf{x}-\mathbf{y}\|_C$. For n large enough such that $\frac{\alpha^n t^n}{n!}<1$, following the generalization of the Banach fixed-point theorem, $\mathbb{T}$ has a unique fixed point in $C([0,T];(L^2[0,1])^n)$ for which

$$\mathbf{x}(t)=\mathbf{x}_o+\int_0^t\Big(\mathbb{A}_1(\tau)\mathbf{x}(\tau)+\mathbf{u}(\tau)\Big)d\tau.\tag{70}$$

Let $h(t,\varepsilon):=\|\mathbb{A}_1(t+\varepsilon)\mathbf{x}(t+\varepsilon)-\mathbb{A}_1(t)\mathbf{x}(t)\|_2$. Then

$$\begin{aligned}
h(t,\varepsilon)&\le\|\mathbb{A}_1(t+\varepsilon)(\mathbf{x}(t+\varepsilon)-\mathbf{x}(t))\|_2\\
&\quad+\|(\mathbb{A}_1(t+\varepsilon)-\mathbb{A}_1(t))\mathbf{x}(t)\|_2\\
&\le\alpha\|(\mathbf{x}(t+\varepsilon)-\mathbf{x}(t))\|_2+\|(\mathbb{A}_1(t+\varepsilon)-\mathbb{A}_1(t))\mathbf{x}(t)\|_2.
\end{aligned}$$

The strong continuity of $\mathbb{A}_1(\cdot)$ and the continuity of $\mathbf{x}(\cdot)$ imply that for any $t\in[0,T]$, $\lim_{\varepsilon\to0}h(t,\varepsilon)=0$. That is $\mathbb{A}_1(\cdot)\mathbf{x}(\cdot)$ is continuous over $[0,T]$. Thus $\mathbb{A}_1(\cdot)\mathbf{x}(\cdot)+\mathbf{u}(\cdot)$ is continuous, and the right-hand side of (70) is differentiable and hence $\mathbf{x}$ is differentiable. Hence (47) has a unique classical solution. ■

Appendix B
Proof of Proposition 8

To prove Proposition 8, we need to introduce a set of two-point boundary value problems and two lemmas. Consider the following homogeneous two-point boundary value (TPBV) problem with a modified time horizon $[t_0,T]$ with $t_0\ge0$:

$$\dot{\mathbf{s}}(t)=-\big[\mathbb{A}(t)^\intercal\big]\mathbf{s}(t)+[(QH-\Pi_tD)\mathbb{I}]\mathbf{z}(t)\tag{71}$$

$$\dot{\mathbf{z}}(t)=[\mathbb{A}(t)+D\mathbf{M}]\mathbf{z}(t)-[BR^{-1}B^\intercal\mathbf{M}]\mathbf{s}(t)\tag{72}$$

where $\mathbf{s}(T) = [Q_T H\mathbb{I}]\mathbf{z}(T) \in (L^2[0,1])^n$ and the modified initial condition is some generic function $\mathbf{z}_{t_0} \in (L^2[0,1])^n$. Define the $\mathbf{M}$-dependent constant

$$\mathrm{L}_{t_0}(\mathbf{M}) \triangleq \sup_{t\in[0,T]} \left\{ \int_{t_0}^{t}\int_{\tau}^{T} \Big\| \Big\{ \phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal \mathbf{M}]\phi_2(\tau,q) [(QH-\Pi_q D)\mathbb{I}] \Big\} \Big\|_{\text{op}} dq d\tau \right\} + \sup_{t\in[0,T]} \left\{ \int_{t_0}^{t} \Big\| \phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal \mathbf{M}]\phi_2(\tau,T)[Q_T H\mathbb{I}] \Big\|_{\text{op}} d\tau \right\}. \tag{73}$$

Since all the terms inside the integration from t_0 to T are non-negative, $L_{t_0}(\mathbf{M})$ is non-increasing with respect to t_0 and in particular $L_{t_0}(\mathbf{M}) \le L_0(\mathbf{M})$. Let $r := 2\|\mathbb{P}(T)\|_{\text{op}} + 1$ and

$$M_T := \sup_{t\in[0,T]} \max\{\|\mathbb{A}^\intercal(t)\|_{\text{op}}, \|(\mathbb{A}(t) + [D\mathbf{M}])\|_{\text{op}}\}.$$

Then let $\tau^* \in (0,T]$ be such that

$$\tau^*\Big(2M_T + 2r\|[BR^{-1}B\mathbf{M}]\|_{\text{op}}\Big) \le \frac{1}{2},$$
$$\tau^*\Big(\|[(QH-\Pi_s D)\mathbb{I}]\|_{\text{op}} + r^2\|[BR^{-1}B\mathbf{M}]\|_{\text{op}} + 2rM_T\Big) \le \|\mathbb{P}(T)\|_{\text{op}} + 1.$$

Lemma 7 (Local Existence of Riccati Mild Solution) *The mild solution to* (44) *exists and is unique in the ball*

$$B_{r,\tau^*} := \{\mathbb{F} \in C_u([T-\tau^*,T]; \mathcal{L}((L^2[0,1])^n) : \|\mathbb{F}\| \le r)\}.$$

PROOF Let (45) be denoted by the fixed point function $\mathbb{P} = \gamma(\mathbb{P})$. For $t \in [T-\tau^*,T]$, and $\mathbf{v} \in (L^2[0,1])^n$, we have

$$\begin{aligned}
&\|\gamma(\mathbb{P}(t))\mathbf{v}\|_2 \\
&= \Big\| \mathbb{P}(T)\mathbf{v} + \int_t^T \Big(\mathbb{A}(\tau)^\intercal \mathbb{P}(\tau) + \mathbb{P}(\tau)(\mathbb{A}(\tau) + [D\mathbf{M}]) \\
&\qquad -\mathbb{P}(\tau)[BR^{-1}B^\intercal\mathbf{M}]\mathbb{P}(\tau) - [(QH-\Pi_\tau D)\mathbb{I}]\Big)\mathbf{v} d\tau \Big\|_2 \\
&\le \Big\{ \|\mathbb{P}(T)\|_{\text{op}} + \tau^*\Big(\|[(QH-\Pi_s D)\mathbb{I}]\|_{\text{op}} \\
&\qquad + r^2\|[BR^{-1}B\mathbf{M}]\|_{\text{op}} + 2rM_T \Big) \Big\} \|\mathbf{v}\|_2 \\
&\le \Big(2\|\mathbb{P}(T)\|_{\text{op}} + 1 \Big)\|\mathbf{v}\|_2 = r\|\mathbf{v}\|_2.
\end{aligned}$$

That is $\gamma(\mathbb{P}(\cdot))$ is a mapping from B_{r,τ^*} to B_{r,τ^*}.

For $\mathbb{P}_1$ and $\mathbb{P}_2$ in B_{r,τ^*}, we obtain

$$\begin{aligned}
\gamma(\mathbb{P}_1)(t)\mathbf{v} - \gamma(\mathbb{P}_2)(t)\mathbf{v} = \int_t^T & \Big(\mathbb{A}(\tau)^\intercal(\mathbb{P}_1(\tau) - \mathbb{P}_2(\tau)) \\
&+ (\mathbb{P}_1(\tau) - \mathbb{P}_2(\tau))(\mathbb{A}(\tau) + [D\mathbf{M}]) \\
&+ (\mathbb{P}_2(\tau) - \mathbb{P}_1(\tau))[BR^{-1}B\mathbf{M}]\mathbb{P}_2(\tau) \\
&+ \mathbb{P}_1(\tau)[BR^{-1}B\mathbf{M}](\mathbb{P}_2(s) - \mathbb{P}_1(\tau))\Big)\mathbf{v} d\tau,
\end{aligned}$$

which implies

$$\begin{aligned}
\|\gamma(\mathbb{P}_1)(t) - \gamma(\mathbb{P}_2)(t)\|_{\text{op}} &\le \tau^*\Big(2M_T + 2r\|[BR^{-1}B\mathbf{M}]\|_{\text{op}}\Big)\|\mathbb{P}_2 - \mathbb{P}_1\| \\
&\le \frac{1}{2}\|\mathbb{P}_2 - \mathbb{P}_1\|.
\end{aligned}$$

Therefore $\gamma(\cdot)$ is $\frac{1}{2}$-contraction in B_{r,τ^*} and there exists a unique mild solution $\mathbb{P}$ in B_{r,τ^*}. ■

Following the same contraction argument as for Lemma 4 in Section V-A, we obtain the following lemma.

Lemma 8 *The TPBV problem defined by* (71) *and* (72) *over* $[t_0,T]$ *admits a unique classical solution pair* $(\mathbf{s},\mathbf{z})$ *if* $L_{t_0}(\mathbf{M}) < 1$. □

Now we proceed to prove Proposition 8 in the following.

PROOF The proof is by contradiction following the idea in the proof of [55, Thm. 12]. Suppose that the operator Riccati equation (44) does not have a mild solution $\mathbb{P} \in C_s([0,T]; \mathcal{L}((L^2[0,1])^n)$ over $[0,T]$. First, we observe that there always exists $\tau^* \in (0,T]$ such that the mild solution to (44) exists over a small interval $[T-\tau^*,T]$ by Lemma 7. Then it can be shown that the non-existence of a mild solution to (44) over $[0,T]$ implies that there is a maximum interval of existence $(t^*,T]$ with $t^* \ge 0$. This further implies that there exists a sequence of strictly decreasing time instances $\{t_k\}_{k=1}^{\infty}$ converging to t^* such that

$$\lim_{t_k \downarrow t_*} \|\mathbb{P}(t_k)\|_{\text{op}} = \infty. \tag{74}$$

(Otherwise, we would have $\lim_{t\downarrow t_*} \|\mathbb{P}(t)\|_{\text{op}} < \infty$. Subsequently, there exists $\varepsilon > 0$ which can be arbitrarily small such that $\sup_{t\in(t_*,t_*+\varepsilon]} \|\mathbb{P}(t)\|_{\text{op}} \le C_p$ for some constant $C_p > 0$ independent of ε. Then by the same proof argument as for Lemma 7, we obtain that a unique solution exists over $[t^* + \varepsilon - \delta,\ t^* + \varepsilon]$ and hence over a closed interval $[t^* + \varepsilon - \delta,\ T] \supset (t^*,T]$, where $\delta > 0$ satisfies

$$\delta > \varepsilon, \quad \delta\Big(2M_T + 2r\|[BR^{-1}B\mathbf{M}]\|_{\text{op}}\Big) \le \frac{1}{2},$$
$$\delta\Big(\|[(QH-\Pi_s D)\mathbb{I}]\|_{\text{op}} + r^2\|[BR^{-1}B\mathbf{M}]\|_{\text{op}} + 2rM_T\Big) \le C_p + 1$$

with $r := 2C_p + 1$. This contradicts the maximum interval of existence $(t^*,T]$.) One can verify that, for $t_k > t^* \ge 0$ (and $[t_k,T] \subset [t_{k+1},T] \subset [t^*,T]$), $L_0(\mathbf{M}) < 1$ implies $L_{t_k}(\mathbf{M}) < 1$ based on the definition of $L_{t_k}(\mathbf{M})$. By Lemma 8, this further implies that the following joint equations have a unique classical solution pair, each of which is in $C([t_k,T]; (L^2[0,1])^n)$:

$$\dot{\mathbf{s}}(t) = -\big[\mathbb{A}(t)^\intercal\big]\mathbf{s}(t) + [(QH-\Pi_t D)\mathbb{I}]\mathbf{z}(t), \tag{75}$$

$$\dot{\mathbf{z}}(t) = [\mathbb{A}(t) + D\mathbf{M}]\mathbf{z}(t) - [BR^{-1}B^\intercal\mathbf{M}]\mathbf{s}(t) \tag{76}$$

with $\mathbf{s}(T) = [Q_T H\mathbb{I}]\mathbf{z}(T) \in (L^2[0,1])^n$ and some generic initial condition $\mathbf{z}_{t_k} \in (L^2[0,1])^n$ with $\|\mathbf{z}_{t_k}\|_2 = 1$. Following arguments similar to those in Section V-A, we obtain that

$$\mathbf{z}(t) = \Gamma_{t_k}(\mathbf{z}) + \phi_1^{\mathbf{M}}(t,0)\mathbf{z}_{t_k} \tag{77}$$

where for $\mathbf{v} \in C([t_k,T]; (L^2[0,1])^n)$,

$$\begin{aligned}
&(\Gamma_{t_k}(\mathbf{v}))(t) \triangleq \\
&-\int_{t_k}^{t} \phi_1^{\mathbf{M}}(t,\tau)[BR^{-1}B^\intercal\mathbf{M}]\Big\{\phi_2(\tau,T)[Q_T H\mathbb{I}]\mathbf{v}(T) \\
&\qquad - \int_{\tau}^{T} \phi_2(\tau,q)\big([(QH-\Pi_q D)\mathbb{I}]\mathbf{v}(q)\big)dq\Big\}d\tau.
\end{aligned}$$

Then by (77),

$$\|\mathbf{z}\|_C \leq \frac{1}{1-L_{t_k}(\mathbf{M})}\|\phi_1^{\mathbf{M}}(t,0)\mathbf{z}_{t_k}\|_2 \leq \frac{K}{1-L_{t_0}(\mathbf{M})},$$

where $K = \sup_{t,\tau\in[0,T]}\|\phi_1^{\mathbf{M}}(t,\tau)\|_{\text{op}}$. In parallel to (33),

$$\begin{aligned}\mathbf{s}(\tau) &= \phi_2(\tau,T)[Q_T H\mathbb{I}]\mathbf{z}(T)\\ &-\int_\tau^T \phi_2(\tau,q)\big([(QH-\Pi_q D)\mathbb{I}]\mathbf{z}(q)\big)dq. \qquad (78)\end{aligned}$$

Hence we can find C_0 independent of t_k and $\mathbf{z}_{t_k}$ such that

$$\sup_{t\in[t_k,T]}\big(\|\mathbf{z}(t)\|_2 + \|\mathbf{s}(t)\|_2\big) \leq C_0. \qquad (79)$$

Following the decoupling technique for TPBV problems in Proposition 5, under the fact that $\mathbb{P}$ exists over $[t_k,T]$, one can verify that the solution pair to (75) and (76) satisfies

$$\mathbf{s}(t) = \mathbb{P}(t)\mathbf{z}(t), \quad \forall t\in[t_k,T]. \qquad (80)$$

We note that the choice of the initial condition $\mathbf{z}_{t_k} \in (L^2[0,1])^n$ with $\|\mathbf{z}_{t_k}\|_2 = 1$ is arbitrary. By (74) and the definition of operator norm, there exists initial conditions $\{\mathbf{z}_{t_k}\}_{k=1}^\infty$ with $\|\mathbf{z}_{t_k}\|_2 = 1$ such that

$$\lim_{k\to\infty}\|\mathbb{P}(t_k)\mathbf{z}_{t_k}\|_2 \geq \lim_{k\to\infty}\Big(\|\mathbb{P}(t_k)\|_{\text{op}} - \frac{1}{k}\Big) = \infty. \qquad (81)$$

Now we take the above $\{\mathbf{z}_{t_k}\}$ as initial conditions for (76). Then by (80), we have $\mathbf{s}(t_k) = \mathbb{P}(t_k)\mathbf{z}_{t_k}$. Hence (81) implies $\lim_{k\to\infty}\|\mathbf{s}(t_k)\|_2 = \infty$, which contradicts (79). Thus we complete the proof. ■

Appendix C
Proof of Proposition 10

PROOF To explicitly verify the eigen pairs of $\mathbf{M}$, we have the following computation. Let $\mathbf{v}_k = \cos\left(\frac{\pi k\beta}{2}\right), k \in \{1,3,5,...\}$. Then for any $\alpha\in[0,1]$ and any $k\in\{1,3,5,...\}$, the following holds

$$\begin{aligned}&[\mathbf{M}\mathbf{v}_k](\alpha) = \int_0^1 (1-\max(\alpha,\beta))\cos\big(\frac{\pi k\beta}{2}\big)d\beta\\ &= \int_0^1 \cos\big(\frac{\pi k\beta}{2}\big)d\beta - \int_0^\alpha \alpha\cos\big(\frac{\pi k\beta}{2}\big)d\beta\\ &-\int_\alpha^1 \beta\cos\big(\frac{\pi k\beta}{2}\big)d\beta = \frac{4}{k^2\pi^2}\cos\big(\frac{\pi k\alpha}{2}\big) = \frac{4}{k^2\pi^2}\mathbf{v}_k(\alpha).\end{aligned}$$

To verify that all the eigen pairs are listed, one can simply check the relation between the sum of squares of the eigenvalues and the 2-norm. First, we compute $\sum_{\ell=1}\lambda_\ell^2 = \big(\frac{4}{\pi^2}\big)^2\sum_{k=0}^\infty\frac{1}{(2k+1)^4} = \frac{1}{6}$. Second, we compute $\|\mathbf{M}\|_2^2 = \int_{[0,1]}\int_{[0,1]}(1-\max(x,y))^2dxdy = \frac{1}{6}$. Therefore, the equality $\|\mathbf{M}\|_2^2 = \sum_{\ell=1}\lambda_\ell^2$ is satisfied, which implies we have listed all the eigenpairs in (65). Hence the spectral decomposition of the uniform attachement graphon limit is then given by (66). ■

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