

Transmission Neural Networks: Inhibitory and Excitatory Connections

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Abstract—This paper extends the Transmission Neural Network model proposed by Gao and Caines in [1]–[3] to incorporate inhibitory connections and neurotransmitter populations. The extended network model contains binary neuronal states, transmission dynamics, and inhibitory and excitatory connections. Under technical assumptions, we establish the characterization of the firing probabilities of neurons, and show that such a characterization in the presence of inhibition can be equivalently represented by a neural network where each neuron has a continuous state of dimension 2. In addition, we incorporate neurotransmitter populations into the modeling and establish the limit network model when the number of neurotransmitters at all synaptic connections goes to infinity. Finally, sufficient conditions for stability and contraction properties of the limit network model are established.

I. INTRODUCTION

Modelling neuronal systems is important to understand intelligence and to analyze and control such systems. Networks of neurons can learn input-output relations in the context of artificial neural networks [4]–[7]. Moreover, recent advances show that combining detailed brain networks, such as the *Drosophila* connectome, with relatively simple neuronal dynamics can predict neural activities associated with specific sensorimotor processing [8]. Neuronal models with different levels of abstraction have been proposed to characterize the behaviors of neuronal systems [9], [10], ranging from the detailed descriptions of the dynamics of individual neurons by Hodgkin and Huxley [11] to network-level models characterizing interactions [9], [12], [13].

Neural network models that adopt a binary-state representation of neuronal systems offer certain advantages: (a) one can focus on network level properties as in the work of Hopfield [12] and those on Boltzmann Machines [14], [15] and (b) continuous-valued neural networks can be binarized to provide more efficient algorithms and learning models [16], [17]. In addition, the binary state for each neuron can be naturally linked with a continuous value by taking the probability of neurons being activated as a neuronal state [1], [13], for which control-theoretic properties including stability can be established (see e.g. [1], [10]).

The Transmission Neural Network (TransNN) model proposed in [1]–[3] has established a natural connection between neural networks and virus spread models, where the

connection of the nodes resembles the process of synaptic transmission. The work [2], [3] investigated how TransNN approximates stochastic neural networks with binary nodal states, and proposed TransNN-based approximate control algorithms for controlling these stochastic networks.

Inhibition that suppresses the activity of neurons is essential for neuronal systems [18]–[20]. Inhibitory properties have been considered in models of neurons with various formulations (e.g. the Wilson-Cowan model for neuronal populations [21], [22] and models with continuous-time stochastic dynamics [23], [24]).

The main focus of the present paper is to generalize TransNN models [2], [3] to account for inhibitory connections. The central challenges are to identify a biologically motivated and mathematically tractable inhibition mechanism and to derive approximate models that capture its effect.

Contribution: This work extends the TransNN models with binary nodal states in [2], [3] to include inhibition, and identifies the corresponding model for the probability of excitation. A key consequence of introducing inhibition is that the scalar-state simplification in [2], [3] no longer applies directly. We show that such networks with inhibition can be equivalently represented by neural networks where a two-dimensional nodal state is associated with each neuron and the Tuneable Log-Sigmoid activation function in [1] with each synaptic connection. Moreover, we incorporate neurotransmitter populations into an extended TransNN model and establish its limit model by letting the number of neurotransmitters at all synaptic connections go to infinity. Finally, sufficient conditions for stability and contraction properties of the limit network model are established.

Notation: $\mathbb{R}$ denotes the set of real numbers. Let $\bar{\mathbb{R}} \triangleq \mathbb{R} \cup \{+\infty\}$ and $[n] \triangleq \{1, 2, \dots, n\}$. We use $W \triangleq [W_{ij}] \in \mathbb{R}^{n \times n}$ to denote the matrix with its ij^{th} element specified by W_{ij} for all $i, j \in [n]$. For a vector $v \in \bar{\mathbb{R}}^n$, we use both v_i and $[v]_i$ to denote its i^{th} element. For a matrix $W \in \mathbb{R}^{n \times n}$, $\|W\|_1 \triangleq \max_{j \in [n]} \sum_{i=1}^n |W_{ij}|$ and $\|W\|_\infty \triangleq \max_{i \in [n]} \sum_{j=1}^n |W_{ij}|$. For a vector $v \in \mathbb{R}^n$, $\|v\|_1 \triangleq \sum_{i=1}^n |v_i|$ and $\|v\|_\infty \triangleq \max_{i \in [n]} |v_i|$.

II. TRANSMISSION DYNAMICS WITH INHIBITORY AND EXCITATORY CONNECTIONS

Consider a network of n neurons with interconnections through chemical synapses [25]. The synaptic connection structure at time or layer¹ $k \geq 0$ is represented by a directed graph $\mathcal{G}^k = ([n], \mathcal{E}^k)$ with the node set $[n] \triangleq \{1, 2, \dots, n\}$

¹We note that, throughout the paper, time step k can also be interpreted as layer k in the context of neural networks with multiple layers.

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and the edge set $\mathcal{E}^k \subset [n] \times [n]$, which may include self-loops. A directed connection from neuron j to neuron i , denoted by the node pair $(i, j) \in \mathcal{E}^k$, exists if the axon terminal of neuron j has at least one synapse onto neuron i at step k . Such synaptic connections could be either excitatory or inhibitory [25]. Let $\mathcal{G}_h^k = ([n], \mathcal{E}_h^k)$ denote the subgraph of $\mathcal{G}^k$ with all inhibitory connections, and $\mathcal{G}_e^k = ([n], \mathcal{E}_e^k)$ the subgraph of $\mathcal{G}^k$ with all excitatory connections. Furthermore, $\mathcal{E}_h^k \cup \mathcal{E}_e^k = \mathcal{E}^k$ and $\mathcal{E}_h^k \cap \mathcal{E}_e^k = \emptyset$, that is, a connection between two neurons will be either inhibitory or excitatory. We allow self-loops to model autapses (i.e. synapses from a neuron onto itself) in neuronal systems [26]. Furthermore, since an autapse is either inhibitory or excitatory [26], the self-loops considered are either inhibitory or excitatory, that is, for each node $i \in [n]$, its self-loop (i.e. the edge pair (i, i)) may appear in either $\mathcal{E}_h^k$ or $\mathcal{E}_e^k$, but not in both. Let B_E^k denote the binary-valued adjacency matrix of the subgraph $\mathcal{G}_e^k = ([n], \mathcal{E}_e^k)$ with all excitatory connections at step k , whose ij^{th} element is 1 if $(i, j) \in \mathcal{E}_e^k$, and 0 otherwise. Similarly, let B_I^k denote the binary-valued adjacency matrix of the subgraph $\mathcal{G}_h^k = ([n], \mathcal{E}_h^k)$ with all inhibitory connections at step k .

A. Transmission Dynamics

The state of a neuron $i \in [n]$ at step k is denoted by a binary variable $X_i(k)$ that takes 1 if the neuron fires and 0 otherwise. We introduce the transmission dynamics with both excitatory and inhibitory connections as follows:

$$X_i(k+1) = \left(1 - \prod_{j \in E_i^{ok}} (1 - W_{ij}^k X_j(k))\right) \times \prod_{j \in I_i^{ok}} (1 - W_{ij}^k X_j(k)) \quad (1)$$

where $E_i^{ok} \triangleq \{j : (i, j) \in \mathcal{E}_e^k\}$ denotes the set of incoming neighbouring nodes of i with excitatory connection that potentially includes node i at step k , $I_i^{ok} \triangleq \{j : (i, j) \in \mathcal{E}_h^k\} = \{j : (i, j) \in \mathcal{E}^k - \mathcal{E}_e^k\}$ denotes the set of incoming neighbouring nodes of i with inhibitory connections that potentially includes node i at step k , and $W_{ij}^k \in \{0, 1\}$ is the binary variable that represents the successful transmission when taking value 1, otherwise 0.

Remark 1 The network model with binary states in (1) generalizes that in [2], [3] to include inhibitory neuronal connections. In the dynamics (1), a single effective inhibitory connection to a neuron suppresses its excitation, whereas in the absence of inhibition, a single effective excitatory connection is sufficient to activate it. This rule provides a coarse-grained representation of inhibitory suppression and is qualitatively reminiscent of the functional effect of shunting inhibition (also known as the short-circuiting effect) [25, p. 290], in the sense that inhibitory input can suppress the effect of excitatory input. $\square$

Remark 2 (Functional Completeness) The existence of a constant input is possible for neuronal systems (e.g. persistent firing for working memory [27]). With a constant input 1 as well as excitatory and inhibitory connections, the

interaction rules in (1) can form an NOR gate with a simple network structure as illustrated in Fig. 1. Since any Boolean

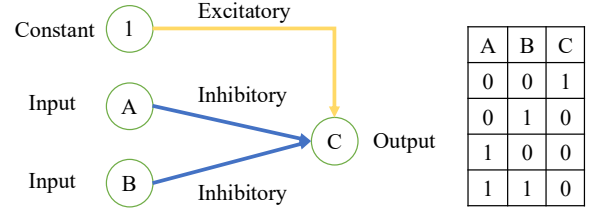


Fig. 1: NOR gate with inputs A and B , and output C , created with inhibitory connections and a constant signal 1.

function can be implemented by combining NOR gates (i.e. the NOR gate is functionally complete), the interaction rule in (1) with the constant signal 1 can reproduce any Boolean function by appropriate network designs. $\square$

B. Stochasticity in the Transmission Dynamics

Now we consider the case where the state $X_i(k)$ and the transmission W_{ij}^k for $i \in [n]$ and for $k \geq 0$ are stochastic. Let $X(k) \triangleq [X_1(k), \dots, X_n(k)]^T$.

We introduce the assumptions on the independence of the transmissions and that of the states.

- (A1) (Memoryless Transmission) For any $k > 0$, $W^k \triangleq [W_{ij}^k]$ is independent of $\{W^t : 0 \leq t < k\}$ and $\{X(t) : 0 \leq t < k\}$. W^0 is independent of $X(0)$.
- (A2) (Transmission Conditional Independence) For any $k \geq 0$, the binary random variables $\{W_{ij}^k : i, j \in [n]\}$ representing the transmissions are jointly conditionally independent given the current states $X(k)$.

Remark 3 The assumption (A1) introduces the independence of the transmission at the current step (or layer) from all the past transmissions and states. (A2) imposes the conditional independence of transmissions across different links given the current state. A special case is as follows: W_{ij}^k is determined by flipping a biased coin independently with probability w_{ij}^k being head corresponding to the value 1, and $X_i(k)$ is determined similarly. $\square$

Under the assumption (A1), the transmission dynamics in (1) are Markovian and

$$\mathbb{E}(X_i(k+1) | (X(t))_{t \in [k]}) = \mathbb{E}(X_i(k+1) | X(k)). \quad (2)$$

Furthermore, under (A2), we have

$$\begin{aligned} \mathbb{E}(X_i(k+1) | X(k)) &= \left(1 - \prod_{j \in E_i^{ok}} \mathbb{E}(1 - W_{ij}^k X_j(k) | X(k))\right) \\ &\quad \times \prod_{j \in I_i^{ok}} \mathbb{E}(1 - W_{ij}^k X_j(k) | X(k)). \end{aligned}$$

For a state configuration $q \in \{0, 1\}^n$, the conditional probability of reaching q is given by

$$\Pr(X(k+1) = q | X(k)) = \prod_{i=1}^n \Pr(X_i(k+1) = q_i | X(k)) \quad (3)$$

where the equality is due to the conditional independence of the transmissions $\{W_{ij}^k : i, j \in [n]\}$ assumed in (A2).

Proposition 1 Assume (A1) and (A2) hold. Given a state configuration $x \in \{0, 1\}^n$ at step k , the transition probability to a state configuration $q \in \{0, 1\}^n$ is given by

$$\Pr(X(k+1) = q | X(k) = x) = \prod_{i=1}^n \left(q_i \rho_i(k+1) + (1 - q_i)(1 - \rho_i(k+1)) \right) \quad (4)$$

where

$$\rho_i(k+1) = \left(1 - \prod_{j \in E_i^{\circ k}} (1 - w_{ij}^k x_j) \right) \times \prod_{j \in I_i^{\circ k}} (1 - w_{ij}^k x_j), \quad i \in [n] \quad (5)$$

and $w_{ij}^k \triangleq \Pr(W_{ij}^k = 1 | X_j(k) = 1)$. $\square$

PROOF Following (3), we note that

$$\Pr(X(k+1) = q | X(k) = x) = \prod_{i=1}^n \Pr(X_i(k+1) = q_i | X(k) = x). \quad (6)$$

Then, by explicitly evaluating each probability $\Pr(X_i(k+1) = q_i | X(k) = x)$, using the dynamics (1), we obtain the desired result. $\blacksquare$

The result above generalizes that of [2, Prop. 1] by including inhibitions in the transmission dynamics.

To further simplify the model, we introduce two other assumptions below.

- (A3) (Transmission Independence at Step k) For $k \geq 0$, $\{W_{ij}^k : i, j \in [n]\}$ are independent and for each $i, j \in [n]$, W_{ij}^k is independent of $\{X_q(k) : q \in [n], q \neq j\}$.
- (A4) (State Independence among Neurons) Up to some terminal step T , for each $k \leq T$, the states $\{X_i(k) : i \in [n]\}$ are independent.

Under (A1), (A3) and (A4), taking the expectation on both sides of the equation (1) yields

$$\mathbb{E}X_i(k+1) = \left(1 - \prod_{j \in E_i^{\circ k}} \mathbb{E}(1 - W_{ij}^k X_j(k)) \right) \times \prod_{j \in I_i^{\circ k}} \mathbb{E}(1 - W_{ij}^k X_j(k)) \quad (7)$$

for every $k \in \{0, \dots, T-1\}$. Let $w_{ij}^k \triangleq \Pr(W_{ij}^k = 1 | X_j(k) = 1)$ denote the conditional probability of the successful transmission from node j to node i at step k . Let

$p_i(k) \triangleq \Pr(X_i(k) = 1)$. Then the equation (7) is equivalently represented by

$$p_i(k+1) = \left(1 - \prod_{j \in E_i^{\circ k}} (1 - w_{ij}^k p_j(k)) \right) \times \prod_{j \in I_i^{\circ k}} (1 - w_{ij}^k p_j(k)). \quad (8)$$

Remark 4 The assumption (A3) introduces the independence of the transmissions across different links and their independence of each link W_{ij}^k from the states of neurons other than its source neuron j , and (A3) is more restrictive than (A2). (A4) can be satisfied with appropriate tree-like network structures and small time horizon T that avoid state correlations. Such structures as well as the characterization of the error incurred in terms of the second moment when (A4) does not hold will be presented in the journal extension. $\square$

C. Dynamics with State Transformation

The state transformation that simplifies the model in [1]–[3] does not apply directly to the model (8) due to the existence of the inhibition probability. To address this issue, we introduce the probability of no inhibition at the previous step $k-1$ (from all neighboring neurons) at neuron i below:

$$\pi_i(k) \triangleq \prod_{j \in I_i^{\circ k-1}} (1 - w_{ij}^{k-1} p_j(k-1)) \in [0, 1]. \quad (9)$$

Then the probability update in (8) is equivalently given by

$$p_i(k+1) = \left(1 - \prod_{j \in E_i^{\circ k}} (1 - w_{ij}^k p_j(k)) \right) \pi_i(k+1) \quad (10)$$

which yields

$$1 - \frac{p_i(k+1)}{\pi_i(k+1)} = \prod_{j \in E_i^{\circ k}} (1 - w_{ij}^k p_j(k)). \quad (11)$$

From (10), clearly $p_i(k) \leq \pi_i(k)$ holds for all $i \in [n]$ and all $k \geq 1$. Define the following states (of Shannon information)

$$s_i(k) \triangleq \begin{cases} -\log\left(1 - \frac{p_i(k)}{\pi_i(k)}\right), & \text{if } \pi_i(k) \in (0, 1] \\ 0, & \text{if } \pi_i(k) = 0 \end{cases} \quad (12)$$

$$o_i(k) \triangleq -\log \pi_i(k) \quad (13)$$

where $o_i(k), s_i(k) \in [0, +\infty]$ and log function is defined by

$$\log(x) \triangleq \begin{cases} \ln(x), & x \in (0, 1]; \\ -\infty, & x = 0. \end{cases} \quad (14)$$

Remark 5 For neuron i , the state $o_i(k)$ is the Shannon information associated with the absence of inhibition from the neighboring neurons at the previous step $k-1$, which can be intuitively understood as the inhibition level. In particular, the state $o_i(k) = 0$ represents no active inhibition at $k-1$ and $o_i(k) = \infty$ represents active inhibition at $k-1$. For a neuron i , its state $s_i(k)$ is essentially the Shannon information

associated with the absence of excitations from the excitatory neighboring neurons at the previous step $k - 1$, that is

$$s_i(k) = -\log\left(1 - \prod_{j \in E_i^{\circ k-1}} (1 - w_{ij}^{k-1} p_j(k-1))\right).$$

The state $s_i(k) = 0$ if the neuron i is resting with probability 1 at time k (i.e. $p_i(k) = 0$), and $s_i(k) = +\infty$ if it fires with probability 1 (i.e. $p_i(k) = 1$). $\square$

Remark 6 The inverse mappings of the state transformations in (13) and (12) are respectively given by

$$\pi_i(k) = e^{-o_i(k)} \quad \text{and} \quad p_i(k) = e^{-o_i(k)}(1 - e^{-s_i(k)}). \quad \square$$

Taking logarithm and negation on both sides of (11) yields the following representation of the evolution of $s_i(k)$

$$s_i(k+1) = \sum_{j \in E_i^{\circ k}} \Psi(w_{ij}^k \pi_j(k), s_j(k)) \quad (15)$$

where $\Psi(w, x) \triangleq -\log(1 - w + we^{-x})$ is the Tuneable Log-Sigmoid (TLogSigmoid) activation function identified in [1]. Replacing $\pi_j(k)$ by $e^{-o_j(k)}$ from the relation (13) yields

$$s_i(k+1) = \sum_{j \in E_i^{\circ k}} \Psi(w_{ij}^k e^{-o_j(k)}, s_j(k)), \quad i \in [n] \quad (16)$$

with initial condition $s_i(0) = -\log(1 - p_i(0))$ for all $i \in [n]$. Furthermore, taking logarithm and negation on both sides of (9) yields dynamics of state o_i as follows:

$$o_i(k+1) = \sum_{j \in I_i^{\circ k}} \Psi(w_{ij}^k e^{-o_j(k)}, s_j(k)), \quad i \in [n] \quad (17)$$

with initial condition $o_i(0) = 0$ for all $i \in [n]$, if there is no inhibition before the initial step.

Equations (16) and (17) then completely characterize the evolution of the state $(s_i(k), o_i(k))$ over time step $k \geq 0$. The evolution can be computed as follows:

- 1) Start with $o_i(0) = 0$ for all $i \in [n]$, if there is no inhibition before step $k = 0$.
- 2) Compute the state $s_i(0) = -\log(1 - p_i(0))$ based on the probability of excitation $p_i(0)$ for all node $i \in [n]$.
- 3) Iteratively compute the states s_i and o_i over time based on the dynamics specified in (16) and (17).

Remark 7 (Convexity) We highlight that if $w \in (0, 1)$ and $x \in (0, \infty)$, the activation function $\Psi(w, x) \triangleq -\log(1 - w + we^{-x})$ is strictly convex in w and strictly concave in x , and is strictly monotonically increasing with respect to x and w by evaluating the derivatives (see [1, Section V]). Let

$$g(z; v, s) \triangleq \Psi(v e^{-z}, s), \quad v \in (0, 1), s \in (0, +\infty)$$

Then $\partial_z g = \partial_w \Psi(v e^{-z}, s) v (-1) e^{-z}$, and

$$\begin{aligned} \partial_{zz}^2 g &= \partial_w \Psi(v e^{-z}, s) v e^{-z} + \partial_{ww}^2 \Psi(v e^{-z}, s) (v (-1) e^{-z})^2 \\ &= v e^{-z} (\partial_w \Psi(v e^{-z}, s) + \partial_{ww}^2 \Psi(v e^{-z}, s) v e^{-z}) > 0 \end{aligned}$$

since $\partial_w \Psi(v e^{-z}, s) > 0$ and $\partial_{ww}^2 \Psi(v e^{-z}, s) > 0$ (see [1, Section V]). This implies that the function $\Psi(v e^{-z}, s)$ is strictly convex in z when $v \in (0, 1), s \in (0, +\infty)$. $\square$

III. MODELS WITH NEUROTRANSMITTER POPULATIONS

To account for different realizations of the effective receptions of different neurotransmitters [25] over the same link, we introduce the following generalized model:

$$\begin{aligned} X_i(k+1) &= \left(1 - \prod_{j \in E_i^{\circ k}} \prod_{\ell=1}^{a_{ij}^k} (1 - W_{ij(\ell)}^k X_j(k))\right) \\ &\times \prod_{j \in I_i^{\circ k}} \prod_{\ell=1}^{a_{ij}^k} (1 - W_{ij(\ell)}^k X_j(k)) \end{aligned} \quad (18)$$

where a_{ij}^k denotes the number of neurotransmitters sent from neuron j to neuron i at step k , and $W_{ij(\ell)}^k$ is a binary variable that represents the successful reception of the ℓ^{th} neurotransmitter at step k from neuron j to neuron i when taking 1, otherwise 0. In this way, the successful reception of a neurotransmitter represented by a binary random variable $W_{ij(\ell)}^k$ is realized at each transmission ℓ at step k from neuron j to neuron i .

We introduce the following assumptions regarding independence of neurotransmissions to identify an approximate model for the probability of excitation.

- (A5)** (Memoryless Neurotransmission) For any $k > 0$, $W^k \triangleq [W_{ij}^k] \in \mathbb{R}^{n \times n \times a_{ij}^k}$ is independent of $\{W^t : 0 \leq t < k\}$ and $\{X(t) : 0 \leq t < k\}$.
- (A6)** (Neurotransmission Conditional Independence) For any $k \geq 0$, the binary random variables $\{W_{ij(\ell)}^k : i, j \in [n], \ell \in [a_{ij}^k]\}$ representing the transmissions are jointly conditionally independent given the current states $X(k)$.

Following similar steps of the previous section, under assumptions (A5) and (A6), we obtain

$$\begin{aligned} \mathbb{E}(X_i(k+1) | (X(t))_{t \in [k]}) &= \mathbb{E}(X_i(k+1) | X(k)) \\ &= \left(1 - \prod_{j \in E_i^{\circ k}} \prod_{\ell=1}^{a_{ij}^k} \mathbb{E}(1 - W_{ij(\ell)}^k X_j(k) | X(k))\right) \\ &\times \prod_{j \in I_i^{\circ k}} \prod_{\ell=1}^{a_{ij}^k} \mathbb{E}(1 - W_{ij(\ell)}^k X_j(k) | X(k)). \end{aligned} \quad (19)$$

Proposition 2 Assume (A5) and (A6) hold. Given a state configuration $x \in \{0, 1\}^n$ at step k , the transition probability to a state configuration $q \in \{0, 1\}^n$ is given by

$$\begin{aligned} \Pr(X(k+1) = q | X(k) = x) \\ = \prod_{i=1}^n \left(q_i \rho_i(k+1) + (1 - q_i)(1 - \rho_i(k+1)) \right) \end{aligned} \quad (20)$$

where

$$\begin{aligned} \rho_i(k+1) &= \left(1 - \prod_{j \in E_i^{\circ k}} (1 - w_{ij}^k x_j)^{a_{ij}^k}\right) \\ &\times \prod_{j \in I_i^{\circ k}} (1 - w_{ij}^k x_j)^{a_{ij}^k}, \quad i \in [n]. \end{aligned} \quad (21) \quad \square$$

The proof is similar to that of Prop. 1 and hence omitted.

Remark 8 Compared to Proposition 1, the difference lies in the representation of the probability $\rho_i(k+1)$ in (21) which now involves the number of neurotransmitters a_{ij}^k with $i, j \in [n]$ and $k \geq 0$.

Such results that evaluate transition probabilities are needed for computing optimal control solutions under the framework of Markov decision processes [2], [3]. $\square$

We introduce the following assumptions to further simplify the representation.

- (A7) (Neurotransmission Independence at Step k) At step $k \geq 0$, $\{W_{ij(\ell)}^k : i, j \in [n], \ell \in [a_{ij}^k]\}$ are independent, and for each $i, j \in [n]$ and each $\ell \in [a_{ij}^k]$, $W_{ij(\ell)}^k$ is independent of $\{X_q(k) : q \in [n], q \neq j\}$.
- (A8) (State Independence among Neurons) Up to some terminal step T , for each $k \leq T$, the underlying binary random variables $\{X_i(k) : i \in [n]\}$ are independent.

Under (A5), (A7) and (A8), taking the expectation on both sides of the equation (18) above yields

$$\begin{aligned} \mathbb{E}X_i(k+1) &= \mathbb{E}\left(1 - \prod_{j \in E_i^{\circ k}} \prod_{\ell=1}^{a_{ij}^k} (1 - W_{ij(\ell)}^k X_j(k))\right) \\ &\times \mathbb{E} \prod_{j \in I_i^{\circ k}} \prod_{\ell=1}^{a_{ij}^k} (1 - W_{ij(\ell)}^k X_j(k)). \end{aligned} \quad (22)$$

Denote $p_i(k) \triangleq \Pr(X_i(k) = 1)$. Let $w_{ij}^k \triangleq \Pr(W_{ij(\ell)}^k = 1 | X_j(k) = 1)$ denote the conditional probability of the successful reception of each neurotransmitter from node j to node i at step k . Then the equation above is equivalent to

$$\begin{aligned} p_i(k+1) &= \left(1 - \prod_{j \in E_i^{\circ k}} (1 - w_{ij}^k p_j(k))^{a_{ij}^k}\right) \\ &\times \prod_{j \in I_i^{\circ k}} (1 - w_{ij}^k p_j(k))^{a_{ij}^k}. \end{aligned}$$

With a slight abuse of notation, define

$$\pi_i(k) \triangleq \prod_{j \in I_i^{\circ k-1}} (1 - w_{ij}^{k-1} p_j(k-1))^{a_{ij}^{k-1}} \in [0, 1] \quad (23)$$

$$s_i(k) \triangleq \begin{cases} -\log\left(1 - \frac{p_i(k)}{\pi_i(k)}\right), & \text{if } \pi_i(k) \in (0, 1] \\ 0, & \text{if } \pi_i(k) = 0 \end{cases} \quad (24)$$

$$o_i(k) \triangleq -\log \pi_i(k) \in [0, +\infty]. \quad (25)$$

with $o_i(k), s_i(k) \in [0, +\infty]$.

Following the same analysis in the previous section, we obtain the following dynamics

$$s_i(k+1) = \sum_{j \in E_i^{\circ k}} a_{ij}^k \Psi(w_{ij}^k e^{-o_j(k)}, s_j(k)), \quad i \in [n] \quad (26)$$

$$o_i(k+1) = \sum_{j \in I_i^{\circ k}} a_{ij}^k \Psi(w_{ij}^k e^{-o_j(k)}, s_j(k)), \quad i \in [n] \quad (27)$$

which explicitly include the numbers of neurotransmitters $\{a_{ij}^k : i, j \in [n], k \geq 0\}$ into the evolution dynamics.

The initial conditions can be given by $o_i(0) = 0$ (if no inhibition exists before the starting time), and $s_i(0) = -\log(1 - p_i(0))$, for all $i \in [n]$.

Remark 9 The equation pair (16) and (17) can now be viewed as special cases of the equation pair (26) and (27) by setting $a_{ij}^k = 1$ for all $i, j \in [n]$ and $k \geq 0$. $\square$

Remark 10 (Biological Interpretation) An important feature of the model above is that the connection weights a_{ij}^k and w_{ij}^k are non-negative and admit direct biological interpretations (i.e. the number of neurotransmitters and the conditional probability of the successful reception of each neurotransmitter, respectively), in contrast to conventional neural networks that allow negative weights [4], [5].

Such dynamics in (26) and (27) are related to graph neural networks [28] where each node has a continuous state (or nodal feature) of dimension two. The state s_i (resp. o_i) summarizes the incoming influence of excitatory (resp. inhibitory) connections. $\square$

Remark 11 An offset in the dynamics (26) and (27) can be created by introducing one node $n_0 \in [n]$ with no incoming links and with the outgoing transmission probability $w_{in_0}^k < 1$, for $i \in [n]$ and $k \geq 0$, and setting its state $s_{n_0}(k)$ to ∞ .

Let $\odot$ denote Hadamard product and introduce

$$A^k = [a_{ij}^k], \quad \Omega^k = [w_{ij}^k], \quad M^k = A^k \odot \Omega^k$$

which are $n \times n$ matrices. Then we have the following upper bounds for the states of (26) and (27).

Proposition 3 (Upper Bound) Let the initial states be given by $s(0) = [s_1(0), \dots, s_n(0)]^\top$ with $s_i(0) = -\log(1 - p_i(0))$ and $o(0) = [o_1(0), \dots, o_n(0)]^\top$. Then the states of (26) and (27) for $k \geq 1$ satisfy that for all $i \in [n]$,

$$s_i(k) \leq [\mathcal{T}_E(k, 0)s(0)]_i \quad (28)$$

$$o_i(k) \leq [(B_I^{k-1} \odot M^{k-1})\mathcal{T}_E(k-1, 0)s(0)]_i \quad (29)$$

with $\mathcal{T}_E(k, 0) \triangleq (B_E^{k-1} \odot M^{k-1}) \dots (B_E^0 \odot M^0)$. $\square$

See Appendix for the proof.

IV. LIMIT MODEL WITH INFINITE NEUROTRANSMITTERS

A. Limit Model via Poisson Approximation

Since the number of released neurotransmitters could be very large (see e.g. [25, Part III, Chp. 11]), the number of receptors at the post-synaptic neurons are relatively moderate and are assumed not to change over a short period of time, the probability of a successful transmission w_{ij} of each neurotransmitter from node j to node i may decrease with respect to the number of neurotransmitters a_{ij} . Hence we introduce the following assumption.

(A9) The probability of transmission w_{ij}^k depends on the number of transmissions a_{ij}^k as follows:

$$w_{ij}^k = \frac{\lambda_{ij}^k}{a_{ij}^k}, \quad \text{with } \lambda_{ij}^k \text{ fixed } \forall k \geq 0, \forall i, j \in [n].$$

Remark 12 (Poisson Approximation) Consider n independent Bernoulli random variables $Z_1, \dots, Z_n$, each of which takes 1 with probability $\frac{\lambda}{n}$. Then the sum $S = \sum_{i=1}^n Z_i$ follows a binomial distribution and hence can be approximated by a Poisson distribution with rate λ . Then $\Pr(\prod_{i=1}^n (1 - Z_i) = 1) = \Pr(S = 0) \approx e^{-\lambda}$. $\square$

Following the Poisson approximation of binomial distributions, if (A9) holds,

$$\Pr(W_{ij(q)}^k X_j(k) = 1) = w_{ij}^k p_j(k) = \frac{\lambda_{ij}^k p_j(k)}{a_{ij}^k}.$$

Applying the Poisson approximation yields

$$\Pr\left(\prod_{q=1}^{a_{ij}^k} (1 - W_{ij(q)}^k X_j(k)) = 1\right) \approx e^{-\lambda_{ij}^k p_j(k)}. \quad (30)$$

Under (A5), (A7), (A8) and (A9), with the Poisson approximation, the expected state then satisfies

$$\begin{aligned} \mathbb{E}X_i(k+1) &= \left(1 - \prod_{j \in E_i^{\circ k}} \mathbb{E} \prod_{q=1}^{a_{ij}^k} (1 - W_{ij(q)}^k X_j(k))\right) \\ &\quad \times \prod_{j \in I_i^{\circ k}} \mathbb{E} \prod_{\ell=1}^{a_{ij}^k} (1 - W_{ij(\ell)}^k X_j(k)) \\ &\approx \left(1 - \prod_{j \in E_i^{\circ k}} e^{-\lambda_{ij}^k p_j(k)}\right) \times \prod_{j \in I_i^{\circ k}} e^{-\lambda_{ij}^k p_j(k)}, \end{aligned} \quad (31)$$

that is

$$p_i(k+1) \approx \left(1 - \prod_{j \in E_i^{\circ k}} e^{-\lambda_{ij}^k p_j(k)}\right) \prod_{j \in I_i^{\circ k}} e^{-\lambda_{ij}^k p_j(k)} \quad (32)$$

for all $i \in [n]$, where $\lambda_{ij}^k = w_{ij}^k a_{ij}^k$ is the rate of the Poisson distribution at time k for the synaptic connection from neuron j to neuron i . Heuristically, this approximation works well when a_{ij}^k is large, w_{ij}^k is small, and $\lambda_{ij}^k = a_{ij}^k w_{ij}^k$ is moderate. To simplify the representation of the dynamics (32), let

$$\bar{\pi}_i(k) \triangleq \prod_{j \in I_i^{\circ k-1}} e^{-\lambda_{ij}^{k-1} p_j(k-1)} \in (0, 1] \quad (33)$$

represent the probability of no inhibition at node i from its neighbouring neurons in the previous step. Let

$$\bar{s}_i(k) \triangleq -\log\left(1 - \frac{p_i(k)}{\bar{\pi}_i(k)}\right) \in [0, +\infty] \quad (34)$$

$$\bar{o}_i(k) \triangleq -\log \bar{\pi}_i(k) \in [0, +\infty]. \quad (35)$$

From (32), we obtain the dynamics for $(\bar{s}, \bar{o})$, given by

$$\bar{s}_i(k+1) = \sum_{j \in E_i^{\circ k}} \lambda_{ij}^k e^{-\bar{o}_j(k)} (1 - e^{-\bar{s}_j(k)}) \quad (36)$$

$$\bar{o}_i(k+1) = \sum_{j \in I_i^{\circ k}} \lambda_{ij}^k e^{-\bar{o}_j(k)} (1 - e^{-\bar{s}_j(k)}). \quad (37)$$

The initial conditions are given by $\bar{o}_i(0) = 0$ (representing no inhibitions before the first step) and $\bar{s}_i(0) = -\log(1 - p_i(0))$, for all $i \in [n]$.

Remark 13 We note that $\bar{\pi}_i(k)$ cannot be zero following its definition in (33), since λ_{ij}^k is assumed to be finite, and $p_j(k-1) \in [0, 1]$. Therefore, we excluded $+\infty$ in (35). $\square$

Proposition 4 Assume (A5), (A7), (A8) and (A9) hold. Then the limit model for the probability of excitation for the dynamics (18), when $a_{ij}^k \rightarrow \infty$ for all $i, j \in [n]$ and $k \in \{0, \dots, T-1\}$, is given by (36) and (37) with

$$\Pr(X_i(k) = 1) = e^{-\bar{o}_i(k)} (1 - e^{-\bar{s}_i(k)}),$$

where $k \in \{0, \dots, T\}$. $\square$

PROOF Under (A5), (A7), (A8) and (A9),

$$\begin{aligned} &\lim_{a_{ij}^k \rightarrow \infty} \mathbb{E} \prod_{q=1}^{a_{ij}^k} (1 - W_{ij(q)}^k X_j(k)) \\ &= \lim_{a_{ij}^k \rightarrow \infty} \Pr\left(\prod_{q=1}^{a_{ij}^k} (1 - W_{ij(q)}^k X_j(k)) = 1\right) \\ &= \lim_{a_{ij}^k \rightarrow \infty} (1 - w_{ij}^k p_j(k))^{a_{ij}^k} \\ &= \lim_{a_{ij}^k \rightarrow \infty} \left(1 - \frac{\lambda_{ij}^k}{a_{ij}^k} p_j(k)\right)^{a_{ij}^k} = e^{-\lambda_{ij}^k p_j(k)}. \end{aligned}$$

that is, the equality in (30) is exact when the number of neurotransmitters goes to infinity on each link. The rest of the proof follows by replacing the approximate equalities in (31) and (32) by exact equalities. $\blacksquare$

To facilitate further analysis, we introduce the element-wise activation function $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined below

$$\phi(s, o) \triangleq [\sigma(s_1, o_1), \dots, \sigma(s_n, o_n)]^\top \in \mathbb{R}^n, \quad \forall s, o \in \mathbb{R}^n$$

with $\sigma(s_i, o_i) \triangleq e^{-o_i} (1 - e^{-s_i})$ for any $s_i, o_i \in \mathbb{R}$. Let $\bar{s}(k) = [\bar{s}_1(k), \dots, \bar{s}_n(k)]^\top$ and $\bar{o}(k) = [\bar{o}_1(k), \dots, \bar{o}_n(k)]^\top$. Then the dynamics in (36) and (37) can be represented in a compact form below

$$\begin{bmatrix} \bar{s}(k+1) \\ \bar{o}(k+1) \end{bmatrix} = \begin{bmatrix} B_E^k \odot \Lambda^k \\ B_I^k \odot \Lambda^k \end{bmatrix} \phi(\bar{s}(k), \bar{o}(k)) \quad (38)$$

where $\Lambda^k \triangleq [\lambda_{ij}^k] \in \mathbb{R}^{n \times n}$ and $\odot$ denotes Hadamard product. We note that the diagonal elements of binary-valued adjacency matrices B_E^k and B_I^k could be non-zero due to the existence of self-loops (representing autapses).

Let $p(k) \triangleq [p_1(k), \dots, p_n(k)]^\top$. For any vector $v \in \mathbb{R}^n$, its exponential function is defined by $e^v = [e^{v_1}, \dots, e^{v_n}]^\top$.

Proposition 5 Assume (A5), (A7), (A8) and (A9) hold. Then the limit model for the probability of excitation for the dynamics (18), when $a_{ij}^k \rightarrow \infty$ for all $i, j \in [n]$ and $k \in \{0, \dots, T-1\}$, is given by

$$p(k+1) = (1 - e^{-(B_E^k \odot \Lambda^k)p(k)}) \odot e^{-(B_I^k \odot \Lambda^k)p(k)} \quad (39)$$

with $\Pr(X_i(k) = 1) = p_i(k)$ for all $i \in [n]$, where $k \in \{0, \dots, T-1\}$. $\square$

PROOF We follow the same proof steps in Prop. 4 to establish an exact equality in (32), for which the compact representation is equivalently given by (39). ■

A trivial equilibrium for (39) is $p^* = [0, \dots, 0]^T \in \mathbb{R}^n$.

B. Contraction and Stability Properties

Proposition 6 (Contraction) *Let $p = 1$ or ∞ . If*

$$\left\| \begin{bmatrix} B_E^k \odot \Lambda^k \\ B_I^k \odot \Lambda^k \end{bmatrix} \right\|_p < 1, \quad \forall k \geq 0 \quad (40)$$

holds, the system in (36) and (37) is contracting, that is,

$$\left\| \begin{bmatrix} \bar{s}(k+1) \\ \bar{o}(k+1) \end{bmatrix} - \begin{bmatrix} \bar{s}^*(k+1) \\ \bar{o}^*(k+1) \end{bmatrix} \right\|_p < \left\| \begin{bmatrix} \bar{s}(k) \\ \bar{o}(k) \end{bmatrix} - \begin{bmatrix} \bar{s}^*(k) \\ \bar{o}^*(k) \end{bmatrix} \right\|_p \quad (41)$$

and

$$\left\| \begin{bmatrix} \bar{s}(k) \\ \bar{o}(k) \end{bmatrix} - \begin{bmatrix} \bar{s}^*(k) \\ \bar{o}^*(k) \end{bmatrix} \right\|_p < \left\| \begin{bmatrix} \bar{s}(0) \\ \bar{o}(0) \end{bmatrix} - \begin{bmatrix} \bar{s}^*(0) \\ \bar{o}^*(0) \end{bmatrix} \right\|_p \quad (42)$$

where $\begin{bmatrix} \bar{s}(k) \\ \bar{o}(k) \end{bmatrix}$ (resp. $\begin{bmatrix} \bar{s}^(k) \\ \bar{o}^*(k) \end{bmatrix}$) denotes the state at step k for the trajectory with the initial value $\begin{bmatrix} \bar{s}(0) \\ \bar{o}(0) \end{bmatrix}$ (resp. $\begin{bmatrix} \bar{s}^*(0) \\ \bar{o}^*(0) \end{bmatrix}$).* □

See Appendix for the proof.

Proposition 7 (Upper Bound) *Let the initial states be given by $\bar{s}(0) = [\bar{s}_1(0), \dots, \bar{s}_n(0)]^T$ with $\bar{s}_i(0) = -\log(1 - p_i(0))$ and $\bar{o}(0) = [\bar{o}_1(0), \dots, \bar{o}_n(0)]^T$. Then the states of the system (36) and (37) for any step $k \geq 1$ satisfy that for all $i \in [n]$,*

$$\bar{s}_i(k) \leq [\Gamma_E(k, 0)\bar{s}(0)]_i \quad (43)$$

$$\bar{o}_i(k) \leq [(B_I^{k-1} \odot \Lambda^{k-1})\Gamma_E(k-1, 0)\bar{s}(0)]_i \quad (44)$$

with $\Gamma_E(k, 0) \triangleq (B_E^{k-1} \odot \Lambda^{k-1}) \dots (B_E^0 \odot \Lambda^0)$. □

PROOF Using the property that $1 - e^{-x} \leq x$ for $x \geq 0$ and then following the same proof steps of Prop. 3 to build the linear dynamical system that provides the upper bounds for the states, we can easily obtain the desired results. ■

Proposition 8 (Stability) *Assume $B_E^k = B_E$, $B_I^k = B_I$ and $\Lambda^k = \Lambda$ are invariant with respect to the step $k \geq 0$. Then the system (36) and (37) is asymptotically and exponentially stable with respect to the step k at the origin if*

$$\max_{i \in [n]} |\lambda_i(B_E \odot \Lambda)| < 1$$

with $\{\lambda_i(B_E \odot \Lambda), i \in [n]\}$ as all the eigenvalues of $B_E \odot \Lambda$. □

PROOF Prop. 7 together with the condition for the stability of discrete-time linear systems implies the desired result. ■

Remark 14 Interestingly, the contraction property in Prop. 6 involves both the excitatory and inhibitory networks, whereas the stability property in Prop. 8 depends only on the excitatory network. For the inhibitory connections to be effective, the corresponding neurons that initiate the inhibitions need to be active, whose activation in the present model is governed by the excitatory connections. Hence the excitatory connections play a primary role in determining the stability properties of the system in Prop. 8. □

V. CONCLUSION

We have generalized the TransNN model to include inhibitions and shown that under technical assumptions, the probability of neuronal excitations in TransNNs with both inhibitory and excitatory connections can be equivalently represented by neural networks where each neuron has a two-dimensional continuous state and each link has the TLogSigmoid activation function. Moreover, neurotransmitter populations were considered in an extended model, and Poisson approximation was applied to establish limit models when the number of neurotransmitters at each link goes to infinity. Sufficient conditions for stability and contraction properties of the limit network model have been established.

Future work should investigate non-trivial equilibria, the integration of the neuronal dynamics for action potentials and spiking sequences, the control of such networks, and the game-theoretic analysis of such dynamics with costs (such as energy and resources for firing).

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APPENDIX PROOF OF PROPOSITION 3

PROOF By the concavity of $\Psi(w, x) \triangleq -\log(1-w+we^{-x})$ in x (see [1, Sec. V]), we have for any $z, z^* \in [-\infty, +\infty]$,

$$\Psi(w, z) \leq \Psi(w, z^*) + \partial_x \Psi(w, z^*)(z - z^*), \quad w \in [0, 1].$$

In particular, taking $z^* = 0$ yields $\partial_x \Psi(w, 0) = w$ and hence $\Psi(w, z) \leq wz$. Applying this inequality to (26) yields

$$s_i(k+1) \leq \sum_{j \in E_i^{ok}} a_{ij}^k w_{ij}^k e^{-o_j(k)} s_j(k) \leq \sum_{j \in E_i^{ok}} a_{ij}^k w_{ij}^k s_j(k).$$

for $o_j(k) \in [0, +\infty]$. That is, the state s is element-wise upper bounded by the state of the discrete-time linear system

$$z(k+1) = (B_E^k \odot M^k)z(k), \quad z(0) = s(0), \quad z(k) \in \mathbb{R}^n,$$

the solution of which is given by $z(k) = \mathcal{T}_E(k, 0)s(0)$. Thus, for all $k \geq 0$, $s_i(k) \leq z_i(k) = [\mathcal{T}_E(k, 0)s(0)]_i$. In addition, from the dynamics (27) of the state o , we obtain similarly

$$o_i(k+1) \leq \sum_{j \in I_i^{ok}} a_{ij}^k w_{ij}^k s_j(k), \quad \forall i \in [n]. \quad (45)$$

Therefore,

$$o_i(k+1) \leq [(B_I^k \odot M^k)s(k)]_i \leq [(B_I^k \odot M^k)z(k)]_i.$$

Replacing $z(k)$ by its solution $\mathcal{T}_E(k, 0)s(0)$ and shifting the time index from $k+1$ to k yield the desired result. ■

APPENDIX PROOF OF PROPOSITION 6

PROOF We follow the contraction analysis for dynamical systems (see e.g. [29], [30]). The gradients of $\phi(s, o)$ satisfy

$$\begin{aligned} \frac{\partial \phi}{\partial s}(s, o) &= \text{diag}(e^{-o_1} e^{-s_1}, \dots, e^{-o_n} e^{-s_n}) \\ \frac{\partial \phi}{\partial o}(s, o) &= \text{diag}(-e^{-o_1}(1 - e^{-s_1}), \dots, -e^{-o_n}(1 - e^{-s_n})) \end{aligned}$$

Then the following hold

$$\left\| \left[\frac{\partial \phi}{\partial s}(s, o) \quad \frac{\partial \phi}{\partial o}(s, o) \right] \right\|_p \leq \max_{i \in [n]} e^{-o_i} \leq 1 \quad (46)$$

for all $s, o \in [0, +\infty]^n$ and $p \in \{1, \infty\}$. Let

$$F^k(s, o) \triangleq \begin{bmatrix} B_E^k \odot \Lambda^k \\ B_I^k \odot \Lambda^k \end{bmatrix} \phi(s, o), \quad s, o \in \mathbb{R}^n.$$

The Jacobian of the system (38) satisfies

$$\begin{aligned} J^k(s, o) &\triangleq \begin{bmatrix} \frac{\partial F^k}{\partial s}(s, o) & \frac{\partial F^k}{\partial o}(s, o) \end{bmatrix} \\ &= \begin{bmatrix} B_E^k \odot \Lambda^k \\ B_I^k \odot \Lambda^k \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial s}(s, o) & \frac{\partial \phi}{\partial o}(s, o) \end{bmatrix}. \end{aligned}$$

Then the submultiplicativity of the induced norms ($\|\cdot\|_1$ and $\|\cdot\|_\infty$), together with (40) and (46), implies

$$\sup_{s, o \in [0, +\infty]^n} \|J^k(s, o)\|_p \leq \left\| \begin{bmatrix} B_E^k \odot \Lambda^k \\ B_I^k \odot \Lambda^k \end{bmatrix} \right\|_p < 1. \quad (47)$$

We introduce the following notational simplification:

$$\begin{aligned} y(k) &\triangleq [\bar{s}(k), \bar{o}(k)]^\top, \quad y^*(k) \triangleq [\bar{s}^*(k), \bar{o}^*(k)]^\top \\ \delta y(k) &\triangleq y(k) - y^*(k), \quad f(y(k)) \triangleq F^k(\bar{s}(k), \bar{o}(k)). \end{aligned}$$

Then $\delta y(k+1) = f(y(k)) - f(y^*(k))$. Define

$$g(\tau) \triangleq f(y^*(k) + \tau(y(k) - y^*(k)))$$

for $\tau \in [0, 1]$. Then $\delta y(k+1) = f(y(k)) - f(y^*(k)) = g(1) - g(0) = \int_0^1 g'(\tau) d\tau$. Furthermore, since

$$\begin{aligned} g'(\tau) &= \frac{\partial}{\partial y} f(y^*(k) + \tau(y(k) - y^*(k)))(y(k) - y^*(k)) \\ &= \frac{\partial}{\partial y} f(y^*(k) + \tau \delta y(k)) \delta y(k), \end{aligned}$$

we obtain $\delta y(k+1) = \int_0^1 \frac{\partial}{\partial y} f(y^*(k) + \tau \delta y(k)) d\tau \delta y(k)$. Let $\bar{s}_\tau(k) \triangleq \bar{s}^*(k) + \tau(\bar{s}(k) - \bar{s}^*(k))$ and $\bar{o}_\tau(k)$ defined similarly. That is, $y^*(k) + \tau \delta y(k) = [\bar{s}_\tau(k), \bar{o}_\tau(k)]^\top$. Then

$$\begin{aligned} \frac{\partial}{\partial y} f(y^*(k) + \tau \delta y(k)) &= J^k(\bar{s}_\tau(k), \bar{o}_\tau(k)) \\ &= \begin{bmatrix} B_E^k \odot \Lambda^k \\ B_I^k \odot \Lambda^k \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial s}(\bar{s}_\tau(k), \bar{o}_\tau(k)) & \frac{\partial \phi}{\partial o}(\bar{s}_\tau(k), \bar{o}_\tau(k)) \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \|\delta y(k+1)\|_p &\leq \int_0^1 \left\| \frac{\partial}{\partial y} f(y^*(k) + \tau \delta y(k)) \right\|_p d\tau \|\delta y(k)\|_p \\ &= \int_0^1 \|J^k(\bar{s}_\tau(k), \bar{o}_\tau(k))\|_p d\tau \|\delta y(k)\|_p < \|\delta y(k)\|_p, \end{aligned}$$

where the last inequality is due to (47). Therefore (41) holds. The inequality in (42) follows by iteratively applying the inequality (41). ■