LQG Graphon Mean Field Games:
Graphon Invariant Subspaces

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Motivation and Background

Mean field couplings $\rightarrow$ **Network couplings** (nonhomegenous, pairwise, random)

Graphon theory: model large graphs and graph limits (Lovász-Szegedy 06', Borgs et al. 08', 12', Lovász 12')

Graphon applications:

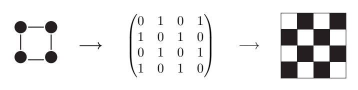
- Dynamical systems: heat equations (Medvedev 14'), coupled oscillators (Chiba-Medvedev 19'), graphon particle systems (Bayraktar-Wu 20', Coppini 21')
- Static games (Parise-Ozdaglar 18', Carmona et al. 19')
- Dynamic games (GMFG Caines-Huang 18',19', 20', Song et. al 20', Carmona et al. 21', etc.)
- Control of large network-coupled dynamical systems (Gao-Caines 17',18',19',20',21')
- Network centrality (Avella-Medina et al. 18'), signal processing (Morency et al. 17'), graph neural networks (Ruiz-Ribeiro-Chamon 19', 20'), epidemic modeling (Gao-Caines 19', Vizuete-Frasca-Garin 20'), etc.

Mean field games on networks: Huang-Caines-Malhamé 10', Guéant 15', Camilli-Marchi 16', Delarue 17', Lacker-Soret 20', Feng-Fouque-Ichiba 20', etc.

Outline

Introduction to Graphons

Graphon Representation of Graphs



Definition (Graphons)

Bounded symmetric Lebesgue measurable functions

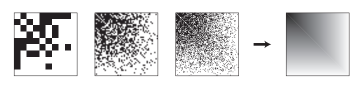
$$\mathbf{W} : [0,1]^2 \to [0,1]$$

interpreted as weighted graphs with the vertex set $[0,1]$.

Notation: $\mathcal{W}_0 := \{\mathbf{W} : [0,1]^2 \to [0,1]\}$ and $\mathcal{W}_c := \{\mathbf{W} : [0,1]^2 \to [-c,c]\}$, $c > 0$

Examples:

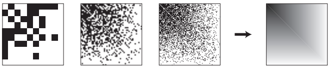
- mean field coupling: $W(x,y) = 1$
- uniform attachment limit: $W(x,y) = 1 - max(x,y)$



Uniform attachment graph sequence converges to the limit under the cut metric w.p. 1 (Lovász 12')

Introduction to Graphons

Compactness of Graphon Space (Lovász 12')



Cut norm: $$\|\mathbf{W}\|_\square := \sup_{S,T\subset[0,1]} \left| \int_{S\times T} \mathbf{W}(x,y)\,dx\,dy \right|$$

Cut metric: $$\delta_\square(\mathbf{W},\mathbf{V}) := \inf_{\phi} \|\mathbf{W}^{\phi} - \mathbf{V}\|_\square,$$

where ϕ is a measure preserving bijections: $\mathbf{W}^{\phi}(x,y) = \mathbf{W}(\phi(x),\phi(y))$.

Theorem (Compactness (Lovász 12'))

The graphon spaces $(\tilde{\mathcal{W}}_0, \delta_\square)$ and $(\tilde{\mathcal{W}}_c, \delta_\square)$ are compact. *

By compactness, infinite sequences of graphons will necessarily possess one or more sub-sequential limits under the cut metric.

* $\tilde{\mathcal{W}}_0$ (resp. $\tilde{\mathcal{W}}_c$) is the space of $\mathcal{W}_0$ (resp. $\mathcal{W}_c$) after identifying equivalent classes of cut distance zero.

Operator $\mathbf{W} : L^2[0,1] \to L^2[0,1]$

$$[\mathbf{W}\mathbf{v}](x) = \int_{[0,1]} \mathbf{W}(x,\alpha)\mathbf{v}(\alpha)d\alpha, \qquad \mathbf{v} \in L^2[0,1], \quad \mathbf{W} \in \mathcal{W}_c$$

Norm relations: $\frac{1}{8}\|\mathbf{W}\|_{\text{op}}^2 \leqslant \|\mathbf{W}\|_{\square} \leqslant \|\mathbf{W}\|_{\text{op}} \leqslant \|\mathbf{W}\|_2$

Operator $[D\mathbf{W}] : \left(L^2[0,1]\right)^n \to \left(L^2[0,1]\right)^n$:

$$([D\mathbf{W}]\mathbf{v})(\alpha) = D\int_{[0,1]} \mathbf{W}(\alpha,\beta)\mathbf{v}(\beta)d\beta, \quad \forall \alpha \in [0,1].$$

where $D \in \mathbb{R}^{n\times n}$, $\mathbf{W} \in \mathcal{W}_c$, and $(L^2[0,1])^n \triangleq \underbrace{L^2[0,1] \times \ldots \times L^2[0,1]}_{n}$.

Spectral Properties of Graphons

Graphon operators are Hilbert-Schmidt operators (and hence compact operators). $\mathbf{M} \in \mathcal{W}_c$ has a countable multi-set of non-zero eigenvalues.

$$\mathbf{M} = \sum_{\ell=1}^{\infty} \lambda_\ell \mathbf{f}_\ell \mathbf{f}_\ell^\top, \quad \text{with } \{\lambda_\ell\} \text{ accumulates at } 0 \text{ and } \sum_{\ell=1}^{\infty} \lambda_\ell^2 = \|\mathbf{M}\|_2^2.$$

where $\{\mathbf{f}_\ell\}$ is the set of orthonormal eigenfunctions

Spectral Decomposition Examples

- Mean Field Coupling: $\mathbf{M}(x, y) = 1, \quad$ (rank-one, $\mathbf{f}_1 = \mathbb{1}$, $\lambda_1 = 1$)
- Step Functions: $\mathbf{M}(x, y) \triangleq \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{1}_{P_i}(x) \mathbb{1}_{P_j}(y) m_{ij}$, (rank($\mathbf{M}$)=rank($M$))
- Uniform Attachment Graphon (Gao-Caines-Huang, arXiv'21):

$$\mathbf{M}(x, y) = 1 - \max(x, y) = \sum_{k=1,3,5,\dots} \frac{4}{k^2\pi^2} \sqrt{2}\cos\left(\frac{k\pi x}{2}\right) \sqrt{2}\cos\left(\frac{k\pi y}{2}\right)$$

Other examples: finite-rank graphons, sinusoidal graphon, idempotent graphon, power-law type graphon ...

Outline

LQG Graphon Mean Field Games: Dynamics

Individual Dynamcis

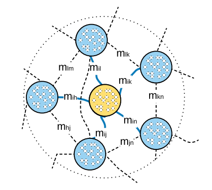
$$dx_i(t) = (Ax_i(t) + Bu_i(t) + Dz_i(t))dt + \Sigma dw_i(t), \quad i \in \{1, ..., K\}$$

- $x_i(t)$, $u_i(t)$, and $z_i(t)$: state, control and network empirical average in $\mathbb{R}^n$;
- $\{w_i, 1 \leqslant i \leqslant K\}$: independent standard n-dimensional Wiener processes.

Network Empirical Average Influence

$$\text{For any } i \in \mathcal{C}_q, \quad z_i(t) = \frac{1}{N} \sum_{\ell=1}^{N} m_{q\ell} \left(\frac{1}{|\mathcal{C}_\ell|} \sum_{j \in \mathcal{C}_\ell} x_j(t) \right)$$

- $\mathcal{C}_q$: set of agents in the q^{th} cluster (node).
- N: total number of such clusters (nodes).
- $M = [m_{q\ell}] \in \mathbb{R}^{N \times N}$: adjacency matrix
- $K = \sum_{q=1}^{N} |\mathcal{C}_q|$: total number of agents.



LQG Graphon Mean Field Games: Cost

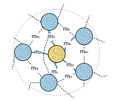
Individual Dynamcis

$$dx_i(t) = (Ax_i(t) + Bu_i(t) + Dz_i(t))dt + \Sigma dw_i(t), \quad i \in \{1, ..., K\}$$

- $x_i(t)$ and $u_i(t)$: state and control in $\mathbb{R}^n$;
- $\{w_i, 1 \leqslant i \leqslant K\}$: independent standard n-dimensional Wiener processes.

Network Empirical Average Influence $z_i(t)$

For any $i \in \mathcal{C}_q$, $\quad z_i(t) = \frac{1}{N}\sum_{\ell=1}^{N} m_{q\ell}\left(\frac{1}{|\mathcal{C}_\ell|}\sum_{j\in\mathcal{C}_\ell} x_j(t)\right)$



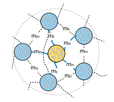
Individual Cost

$$J_i(u_i, u_{-i}) \triangleq \mathbb{E}\int_0^T \left(\|x_i(t) - \nu_i(t)\|_Q^2 + \|u_i(t)\|_R^2\right)dt + \mathbb{E}\|x_i(T) - \nu_i(T)\|_{Q_T}^2$$

- where $\nu_i(t) \triangleq H(z_i(t) + \eta) \in \mathbb{R}^n$
- $Q, Q_T \geqslant 0, R > 0,$

LQG Graphon Mean Field Games

Nodal Population Limit + Network (Gao-Caines-Huang CDC'21)



Taking the local population limit (i.e. $|\mathcal{C}_q| \to \infty$ for all $q \in \{1, ..., N\}$)

$$dx_\alpha(t) = (Ax_\alpha(t) + Bu_\alpha(t) + Dz_\alpha(t))dt + \Sigma dw_\alpha(t), \quad \alpha \in \mathcal{C}_q.$$

$$J_\alpha(u_\alpha, \nu_\alpha) = \mathbb{E}\int_0^T (\|x_\alpha(t) - \nu_\alpha(t)\|_Q^2 + \|u_\alpha(t)\|_R^2)dt + \mathbb{E}\|x_\alpha(T) - \nu_\alpha(T)\|_{Q_T}^2$$

where $\nu_\alpha(t) \triangleq H(z_\alpha(t) + \eta)$.

Network Mean Field Influence $z_\alpha(t)$

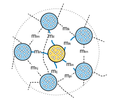
For agent $\alpha \in \mathcal{C}_q$, $\quad z_\alpha(t) = \frac{1}{N}\sum_{\ell=1}^{N} m_{q\ell}\bar{x}_\ell(t), \quad \bar{x}_\ell(t) \triangleq \lim_{|\mathcal{C}_\ell|\to\infty} \frac{1}{|\mathcal{C}_\ell|}\sum_{j\in\mathcal{C}_\ell} x_j(t)$

Best Response (based on LQG Tracking Solution)

$$u_\alpha(t) = -R^{-1}B^\intercal(\Pi_t x_\alpha(t) + \bar{s}_q(t)), \qquad \alpha \in \mathcal{C}_q$$
$$-\dot{\Pi}_t = A^\intercal\Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\intercal\Pi_t + Q, \qquad \Pi_T = Q_T, \tag{1}$$

Forward-Backward Joint Equations on Networks

Nodal Population Limit + Network (Gao-Caines-Huang CDC'21)



Forward Equation: (nN dim)

$$\dot{\bar{z}}(t) = I_N \otimes \left(A - BR^{-1}B^{\intercal}\Pi_t\right)\bar{z}(t) + \frac{1}{N} M \otimes D\bar{z}(t) - \frac{1}{N} M \otimes BR^{-1}B^{\intercal}\bar{s}(t)$$
$$\bar{z}(0) = \frac{1}{N}(I_N \otimes M)\bar{x}(0), \tag{2}$$

Backward Equation: (nN dim)

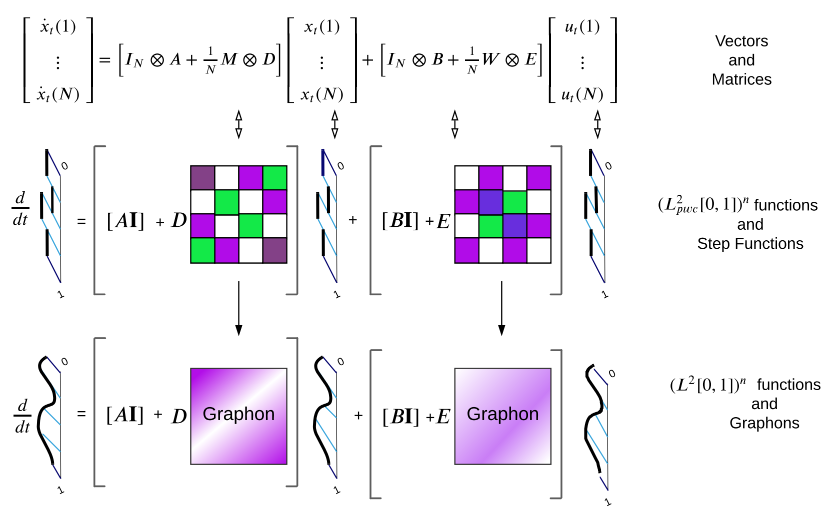
$$-\dot{\bar{s}}(t) = I_N \otimes \left(A - BR^{-1}B^{\intercal}\Pi_t\right)^{\intercal}\bar{s}(t) - I_N \otimes (QH - \Pi_t D)\bar{z}(t) - I_N \otimes Q\gamma\eta$$
$$\bar{s}(T) = H(\bar{z}(T) + \eta), \tag{3}$$

where $I_N \in \mathbb{R}^{N\times N}$ identity matrix, $\bar{z}(t) \triangleq (\bar{z}_1(t)^{\intercal}, \dots, \bar{z}_N(t)^{\intercal})^{\intercal}$, and $\bar{s}(t)$ and $\bar{x}(t)$ are defined similarly.

Solution Complexity

1. the exact network structure and weights!
2. solutions to two coupled nN dimensional equations!

Graphon Dynamical System Approx (Gao-Caines TAC'20, TCNS'21)



Compactness of graphon space ensures graphon limits exist (LL 21')

$L^2_{pwc}[0,1]$: piece-wise constant functions in $L^2[0,1]$ with uniform partition.

Limit Graphon Forward-Backward Joint Equations

Graphon Forward Equation $\quad \mathbf{z}\mathrm{Dyn}(\mathbf{s})$

$$\dot{\mathbf{z}}(t) = \Big([(A - BR^{-1}B^\intercal \Pi_t)\mathbb{I}] + D\mathbf{M}\Big)\mathbf{z}(t) - [BR^{-1}B^\intercal \mathbf{M}]\mathbf{s}(t)$$
$$\mathbf{z}(0) = [\mathbb{I}\mathbf{M}]\bar{x}(0) = \int_{[0,1]} \mathbf{M}(\cdot, \beta)\bar{x}_\beta(0)\,d\beta, \quad \mathbf{z}(t) \in (L^2[0,1])^n \tag{4}$$

(Graphon) Backward Equations $\quad \mathbf{s}\mathrm{Dyn}(\mathbf{z})$

$$\dot{\mathbf{s}}(t) = -\Big([(A - BR^{-1}B^\intercal \Pi_t)\mathbb{I}]^\intercal\Big)\mathbf{s}(t) + [(QH - \Pi_t D)\mathbb{I}]\mathbf{z}(t) + [QH\mathbb{I}]\eta$$
$$\mathbf{s}(T) = [QH_T\mathbb{I}](\mathbf{z}(T) + \eta), \qquad \mathbf{s}(t) \in (L^2[0,1])^n; \tag{5}$$

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Graphon Forward Equation $\mathbf{z}\mathrm{Dyn}(\mathbf{s})$

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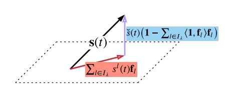
Answers to questions (Gao-Caines-Huang CDC'21, arXiv'21)

(a) Existence and uniqueness of solution pair $(\mathbf{z},\mathbf{s})$?
Contraction condition in $C([0,T];(L^2[0,1])^n)$ with uniform norm $\|\cdot\|_C$.

(b) Asymptotic between of $(\mathbf{z}^{[N]},\mathbf{s}^{[N]})$ to $(\mathbf{z},\mathbf{s})$?
$$\|\mathbf{s} - \mathbf{s}^{[N]}\|_C = O\left\{\max(\|\mathbf{M} - \mathbf{M}^{[N]}\|_{op},\ \|\mathbf{z}(0) - \mathbf{z}^{[N]}(0)\|_2)\right\}$$

Note: $(\mathbf{z}^{[N]},\mathbf{s}^{[N]})$ denotes the piece-wise constant function representation of $(\bar{z},\bar{s})$ in $C([0,T];(L^2_{pwc}[0,1])^n)$

Method 1: Subspace Decomposition of Joint Equations



Project $\mathbf{s}, \mathbf{z}$ into $\mathcal{S}^n$ and $(\mathcal{S}^\perp)^n$, with $\mathcal{S} \triangleq \text{span}\{\mathbf{f}_\ell\}_{\ell \in \mathcal{I}_\lambda}$

Proposition (Gao-Caines-Huang CDC'21)

If Forward-Backward Eqn. (4) *and* (5) *have a unique classical solution pair* $(\mathbf{z}, \mathbf{s})$, *then*

$$(\text{dim } n): \quad \mathbf{s}_\theta(t) = \sum_{\ell \in \mathcal{I}_\lambda} \mathbf{f}_\ell(\theta) s^\ell(t) + \breve{s}(t)\Big(1 - \sum_{\ell \in \mathcal{I}_\lambda} \langle \mathbf{f}_\ell, \mathbf{1} \rangle \mathbf{f}_\ell(\theta)\Big)$$

$$(\text{dim } n): \quad \mathbf{z}_\theta(t) = \sum_{\ell \in \mathcal{I}_\lambda} \mathbf{f}_\ell(\theta) z^\ell(t),$$

for almost all $\theta \in [0,1]$, *for all* $t \in [0,T]$

where z^ℓ, s^ℓ *and* $\breve{s} \in C([0,T]; \mathbb{R}^n)$ *are given by*

$$(\text{dim } n): \quad \dot{s}^\ell(t) = -A_c(t)^\intercal s^\ell(t) + (QH - \Pi_t D) z^\ell(t) + QH\eta, \ s^\ell(T) = \ Q_T H(z^\ell(T) + \eta),$$

$$(\text{dim } n): \quad \dot{z}^\ell(t) = (A_c(t) + \lambda_\ell D) z^\ell(t) - \lambda_\ell B R^{-1} B^\intercal s^\ell(t), z^\ell(0) = \lambda_\ell \int_{[0,1]} \mathbf{f}_\ell(\beta) \bar{x}_\beta(0) d\beta$$

$$(\text{dim } n): \quad \dot{\breve{s}}(t) = -A_c(t)^\intercal \breve{s}(t) + QH\eta, \quad \breve{s}(T) = Q_T H\eta,$$

with $A_c(t) \triangleq (A - BR^{-1}B^\intercal \Pi_t)$.

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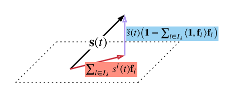

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$$(\dim n): \quad z_\theta(t) = \sum_{\ell\in\mathcal{I}_\lambda} f_\ell(\theta) z^\ell(t), \quad \textit{for almost all } \theta \in [0,1], \textit{ for all } t \in [0,T]$$

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$$(\dim n): \quad \dot{\breve{s}}(t) = -A_c(t)^\intercal \breve{s}(t) + QH\eta, \quad \breve{s}(T) = Q_T H\eta, \qquad \text{with } A_c(t) \triangleq (A - BR^{-1}B^\intercal \Pi_t).$$

Complexity: d forward-backward equation pairs (n-dim) and 1 ODE (n-dim)

d : number of distinct non-zero eigenvalues of graphon $\mathbf{M}$

Method 2: Solution based on Operator Riccati Eqn.

Operator Riccati Equation

$$-\dot{\mathbb{P}} = \mathbb{A}(t)^\intercal \mathbb{P} + \mathbb{P}\mathbb{A}(t) + \mathbb{P}[D\mathbf{M}] - \mathbb{P}[BR^{-1}B^\intercal \mathbf{M}]\mathbb{P} - [(QH - \Pi_t D)\mathbb{I}], \quad \mathbb{P}(T) = [QH_T\mathbb{I}] \tag{6}$$

where $\mathbb{A}(t) = (A - BR^{-1}B^\intercal \Pi_t)\mathbb{I}$ and Π is the solution to (1).

(A2) The operator Riccati equation (6) has a unique mild solution*.

Sufficient Condition for Existence and Uniqueness $(\mathbf{z}, \mathbf{s})$ (Gao-Caines-Huang CDC'21)

Under (A2), joint equations ($\mathbf{z}$Dyn, $\mathbf{s}$Dyn) have a unique classical solution pair $(\mathbf{z}, \mathbf{s})$.

Features of Operator Ricc. Eqn. (Gao-Caines-Huang CDC'21)

- Operator Riccati equation decouple joint equations ($\mathbf{z}$Dyn, $\mathbf{s}$Dyn)
- (A2) is less restrictive than the contraction condition for ($\mathbf{z}$Dyn, $\mathbf{s}$Dyn)

*That is, $\mathbb{P} \in C_s([0,T]; \mathcal{L}((L^2[0,1])^n))$, $P(T) = [Q_T H\mathbb{I}]$, and for all $\mathbf{v} \in (L^2[0,1])^n$,
$$\mathbb{P}(t)\mathbf{v} = \mathbb{P}(T)\mathbf{v} + \int_t^T \Big(\mathbb{A}(\tau)^\intercal \mathbb{P}(\tau) + \mathbb{P}(\tau)(\mathbb{A}(\tau) + [D\mathbf{M}]) - \mathbb{P}(\tau)[BR^{-1}B^\intercal \mathbf{M}]\mathbb{P}(\tau) - [(QH - \Pi_\tau D)\mathbb{I}]\Big)\mathbf{v}\, d\tau.$$

Corollary (Gao-Caines-Huang CDC'21)

If (A2) *holds, then the solution to the operator Riccati equation* (6) *is given by*

$$\mathbb{P}(t) = \Big[P^{\perp}(t)\mathbb{I}\Big] + \sum_{\ell\in\mathcal{I}_\lambda}\Big[\Big(\bar{P}^{\ell}(t) - P^{\perp}(t)\Big)\mathbf{f}_\ell\mathbf{f}_\ell^{\intercal}\Big], \qquad t\in[0,T] \tag{7}$$

$$\mathrm{dim}(n\times n) \qquad -\dot{P^{\perp}} = A_c(t)^{\intercal}P^{\perp} + P^{\perp}A_c(t) - (QH - \Pi_t D), \quad P^{\perp}(T) = QH_T$$

$$\mathrm{dim}(n\times n) \qquad -\dot{\bar{P}}^{\ell} = A_c(t)^{\intercal}\bar{P}^{\ell} + \bar{P}^{\ell}(A_c(t) + \lambda_\ell D) - \lambda_\ell \bar{P}^{\ell}BR^{-1}B^{\intercal}\bar{P}^{\ell}$$
$$-(QH - \Pi_t D), \quad \bar{P}^{\ell}(T) = Q_T H, \quad \ell\in\mathcal{I}_\lambda.$$

$\mathcal{I}_\lambda$: *the index multi-set of non-zero eigenvalues.* $A_c(t) := (A - BR^{-1}B^{\intercal}\Pi_t)$.

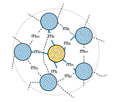
Complexity

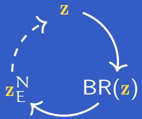
1 ODE $(n\times n)$ and d Riccati equations $(n\times n)$

d : number of distinct non-zero eigenvalues of graphon $\mathbf{M}$

LQG-GMFG Performance Analysis

Asymptotic Error $\|\mathbf{z} - \mathbf{z}_E^N\|_C$





Theorem (Network Empirical Average to Graphon MF, Gao-Caines-Huang CDC'21, arXiv'21)

Assume initial conditions at node $q \in \mathcal{V}_c$ has mean μ_q and uniformly bounded variance. Under the mild technical assumptions the error between the network empirical average $\mathbf{z}_E^N$ and the graphon mean field $\mathbf{z}$ satisfies

$$\mathbb{E}\|\mathbf{z}_E^N - \mathbf{z}\|_C = O\left\{ \max\left(\|\mathbf{M} - \mathbf{M}^{[N]}\|_{op},\ \|\mathbf{z}(0) - \mathbf{z}^{[N]}(0)\|_2,\ \frac{1}{\sqrt{\min_{q \in \mathcal{V}_c} |\mathcal{C}_q|}} \right) \right\}, \tag{8}$$

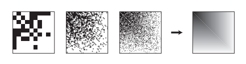
where $\mathbf{z}^{[N]}(0)$ in $(L^2_{pwc}[0,1])^n$ is the piece-wise constant function representation of the initial condition of the network mean field $\bar{z}(0) = \frac{1}{N} M[\mu_1, ..., \mu_N]^\intercal$.

Note: $\|\cdot\|_C$ denotes the uniform norm for $C([0,T];(L^2[0,1])^n)$.

For results with explicit rate of convergence, see (Gao-Caines-Huang arXiv'21).

Numerical Example 1

Uniform Attachement Graphs



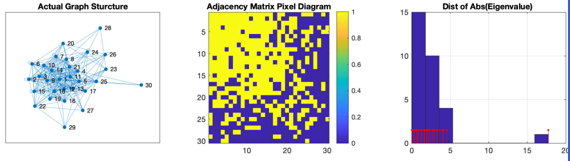


Figure: A random graph instance with 30 nodes generated following the uniform attachment procedure, its pixel representation and the distribution of modulus of the eigenvalues.

Spectral Decomp. of Uniform Attachment Graphon (Gao-Caines-Huang, arXiv'21)

$$\mathbf{M}(x,y) = 1 - \max(x,y) = \sum_{k=1,3,5,\dots} \frac{4}{k^2\pi^2}\sqrt{2}\cos\left(\frac{k\pi x}{2}\right)\sqrt{2}\cos\left(\frac{k\pi y}{2}\right)$$

Approx error by 5 most significant eigendirections: $\approx 1\%$ in $\|\cdot\|_{op}$

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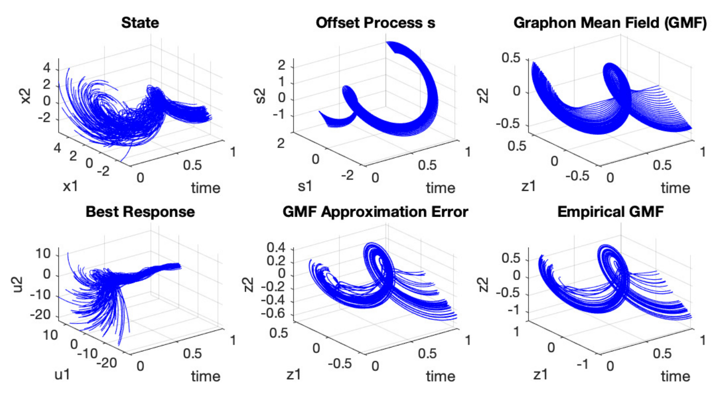

Figure: Simulations on the uniform attachment graph example with 30 nodes where each node contains 4 agents and each agent has 2 states.

LQG-GMFG Parameters: $A = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $B = D = R = Q_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$\eta = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ T = 1, n = 2, \ N = 30, \ |C_\ell| = 4, \ 1 \leqslant \ell \leqslant N. \tag{9}$$

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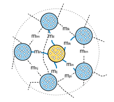

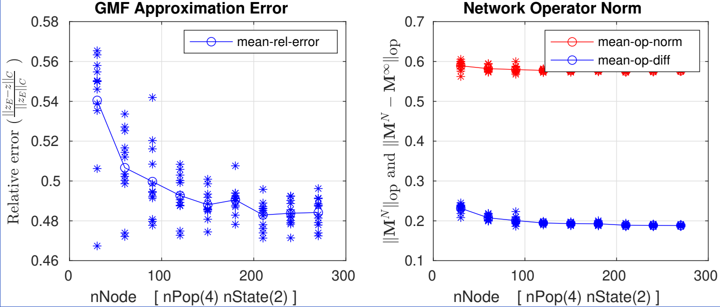

Figure: The relative error in the graphon mean field decreases as graph sizes increase. 12 simulation independent experiments are carried out for each size. The nodal population size denoted by nPop is 4, the local state dimension denoted by nState is 2. In the figure on the right, black dots represent the values for $\|\mathbf{M}^{[N]} - \mathbf{M}\|_{\text{op}}$ in different simulation experiments.

Numerical Example 2

Random Graphs Sampled from SBM

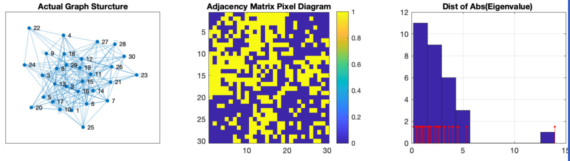

Figure: A graph generated from SBM, its pixel diagram and the distribution of the modulus of eigenvalues.

The block matrix of SBM is given by

$$W = \begin{bmatrix} 0.25 & 0.5 & 0.2 \\ 0.5 & 0.35 & 0.7 \\ 0.2 & 0.7 & 0.4 \end{bmatrix}. \tag{10}$$

Step Function Graphon:

$$\mathbf{M}(x,y) = \sum_{i=1}^{3}\sum_{j=1}^{3} w_{ij} \mathbb{1}_{P_i}(x) \mathbb{1}_{P_j}(y), \quad (x,y) \in [0,1]^2$$

$\text{rank}(\mathbf{M}) = \text{rank}([w_{ij}]) = 3.$

Numerical Example 2

Random Graphs Sampled from SBM

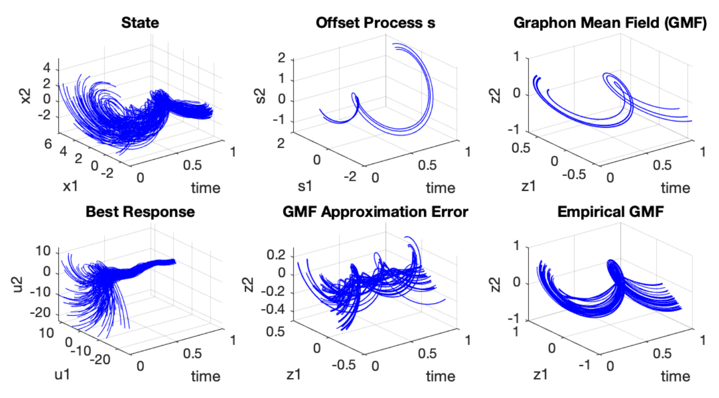

Figure: Simulation on a network generated from SBM with 30 nodes where each node contains 4 agents and each agent has 2 states.

LQG-GMFG Parameters:

$$A = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix}, \ Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, B = D = R = Q_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\eta = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ T = 1, n = 2, \ N = 30, \ |C_\ell| = 4, \ 1 \leqslant \ell \leqslant N. \tag{11}$$

Numerical Example 2

Random Graphs Sampled from SBM

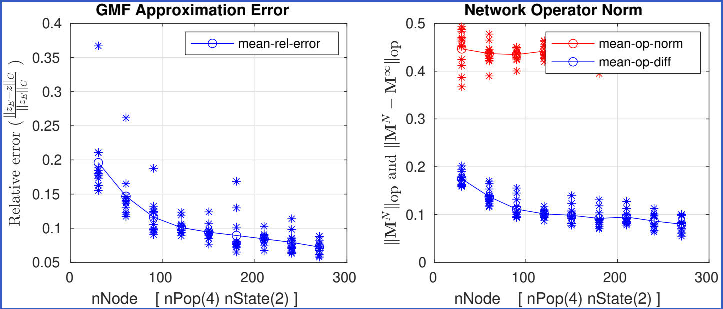

Figure: Graphon mean field game approximation errors on networks of different sizes. 12 simulations are carried out for each size. The nodal population size denoted by nPop is 4, and the local state dimension denoted by nState is 2. In the figure on the right, black dots represent values for $\|\mathbf{M}^{[\mathbf{N}]} - \mathbf{M}\|_{\text{op}}$ in different simulation experiments.

Conclusion and Future Directions

Conclusion

- Subspace decompositions for solving LQG graphon mean field games.
- Sufficient conditions for the existence of a unique LQG-GMFG solution
- Asymptotic rate of approximation errors

Future directions

- Solution methods for nonlinear problems
- General node embedding spaces and embedding mechanism
- Network models with local + neighbourhood + global influence
- Heterogenous local dynamics

Thank You! Questions!

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