Centrality-Weighted Opinion Dynamics:

Disagreement and Social Network Partition

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Motivation: How to identify “influence network”?

“Influence Network” in opinion dynamics:

(French Jr [1956], DeGroot [1974], Friedkin and Johnsen [1990], and many variants)

Not necessarily the underlying network connection structure!

Heterogenous attentions due to nodal properties:

- News/posts on online platforms (ranks by recommender sys.)
- Opinions of individuals (number of followers)
- Attention to research papers (citation counts)

Procedure to identify "influence network"

Social structure + Relevant centralities $\rightarrow$ Influence network

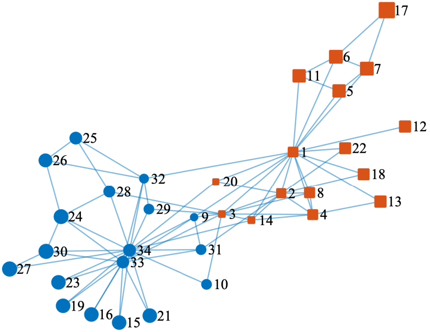
Definition (Network Centrality)

A mapping $\phi : V \rightarrow \mathbb{R}_+$ that quantify how central (or influential) nodes are in a network, where V is the node set.

One relevant centrality in social networks: Degree centrality

Zachary's Karate Club Network (Zachary [1977])

Social interactions among Karate club members. Conflict between node 34 (adm) and node 1(ins) split the group into two groups.



MaxFlowMinCut: Zachary [1977] all but one member (node 9)
Modularity: Newman [2006] correctly characterizes all nodes

Spectral Partition (no degree weights): All but one member (node 3)

Spectral Partition (with degree weights): Gao CDC'21 correctly characterizes all nodes

Outline

1. Degree-Weighted Opinion Dynamics
2. Partition Algorithm and Applications
3. General Centrality-Weighted Opinion Dynamics
4. Conclusion and Future Directions

Basic Modelling Assumptions

(i) **Social Conformity:**
Individuals in a social network communicate and change their own opinions in the direction to conform with those of their neighbours;

(ii) **Degree Weighted Influence:**
Each individual weights these influences from the neighbours promotional to their connection degrees.

Degree-Weighted Opinion Dynamics (Gao CDC'21)

Opinion evolution over a social network:

$$\tau \dot{x}_i = \sum_{j \in N_i} \frac{d_j}{\sum_{j \in N_i} a_{ij} d_j} a_{ij}(x_j - x_i), \quad x_i(0) = x_{i0}, \; i \in [n] \qquad (1)$$

- N_i: the set of neighborhood of node i
- $d_i = \sum_{j \in N_i} a_{ij}$: the degree (centrality) of node i on the network
- a_{ij}: social connection between nodes i and j
- $\tau > 0$ is a fixed time constant.

The new influence matrix: $\bar{A}_{ij} = \frac{d_j}{\sum_{j \in N_i} a_{ij} d_j} a_{ij}$

Degree-Weighted Opinion Dynamics (Gao CDC'21)

Denote $x = [x_1, \ldots, x_n]^\top$. Then

$$\tau \dot{x} = -\bar{L}x, \quad x(0) = x_0. \tag{2}$$

where Laplacian matrix $\bar{L}$ is

$$\bar{L} = I_n - \bar{A}, \quad \text{with } \bar{A} = [\text{diag}(Ad)]^{-1} A \text{diag}(d),$$

Different from normalized Laplacian matrices

$$L_n \triangleq [\text{diag}(d)]^{-1}(\text{diag}(d) - A) = I_n - [\text{diag}(d)]^{-1}A,$$
$$L_{sn} \triangleq I_n - [\text{diag}(d)]^{-\frac{1}{2}} A [\text{diag}(d)]^{-\frac{1}{2}}.$$

Note: $\bar{L}$ is not necessarily symmetric.

Spectral Properties of $\bar{L}$ and $\bar{A}$ (Gao CDC'21)

Assume the underlying graph $\mathcal{G}(A)$ with the adjacency matrix A is connected and undirected.

Properties

(P1): All the eigenvalues of $\bar{A}$ and $\bar{L}$ are real.

(P2): $\bar{A}$ and $\bar{L}$ are diagonalizable.

(P3): $\bar{L}$ contains only one zero eigenvalue and all the other eigenvalues of $\bar{L}$ are strictly positive.

Laplacian matrix $\bar{L}$:

$$\bar{L} = I_n - \bar{A}, \quad \text{with } \bar{A} = [\text{diag}(Ad)]^{-1} A \text{diag}(d).$$

Explicit Solutions and Disagreement State

By (P1)&(P3), we can list eigenvalues of $\bar{L}$ by $0 = \lambda_1 < \lambda_2 \leqslant ... \leqslant \lambda_n$.

By (P2) (i.e. $\bar{L}$ is diagonalizable), the opinion evolution (1) is explicit:

$$x(t) = \sum_{i=1}^{n} e^{-\frac{t}{\tau}\lambda_i} u_i (v_i^T x_0) \quad (3)$$

(λ_i, v_i, u_i): (eigenvalue, left-eigenvector, right-eigenvetor)

Note that $u_1 = \frac{1}{\sqrt{n}}\mathbf{1}$ lies in the agreement subspace $\text{span}(\mathbf{1})$.

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Disagreement state

$$x^{\text{dis}}(t) = \sum_{i=2}^{n} u_i (v_i^T x(t)) = \sum_{i=2}^{n} e^{-\frac{t}{\tau}\lambda_i} u_i (v_i^T x_0). \tag{4}$$

- The slowest rate of exponential decay[1] is governed by $\lambda_2(\bar{L})$ of $\bar{L}$.
- Approximate x^{dis} by the eigen triple: $(\lambda_2(\bar{L}), v_2, u_2)$.

[1] (P3): $\bar{L}$ contains only one zero eigenvalue and all other eigenvalues of $\bar{L}$ are strictly positive.

Outline

Partition Alg. (Social Choice Alg.) (Gao CDC'21)

(S1) If $\lambda_2(\bar{L})$ has algebraic multiplicity 1, let

$$s \triangleq u_2.$$

If $\lambda_2(\bar{L})$ has algebraic multiplicity m_2 $(m_2 \geqslant 2)$, let

$$s \triangleq \sum_{\ell=1}^{m_2} u_2^\ell ({v_2^\ell}^T x_0)$$

$\{(\lambda_2(\bar{L}), v_2^\ell, u_2^\ell,)\}_{\ell=1}^d$: $\{$(eigenval, left-eigenvec, right-eigenvec)$\}_{\ell=1}^d$ and x_0 denotes the initial opinion vector.

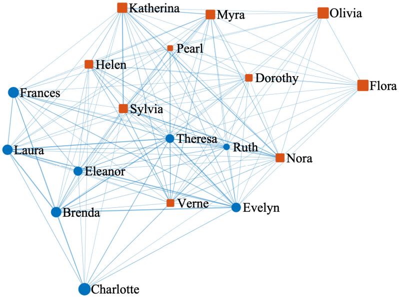
(S2) The signs of elements in $s \in \mathbb{R}^n$ separate the nodes in the network into two clusters as follows:

$$C_1 = \{i : s(i) > 0\}, \qquad C_2 = \{i : s(i) \leqslant 0\}.$$

Note: when $\lambda_2(\bar{L})$ has algebraic multiplicity 1 this is essentially the Fielder spectral partition for the new "influence network" $\mathcal{G}(\bar{A})$.

Applications to Southern Women Network (Davis et al. [1941])

18 women attended 14 events and the connections among them are characterized by the number of co-attended events.



Our algorithm achieves the same bipartition result except one node (node Pearl) as those in Davis et al. [1941] and Liebig and Rao [2014].

In contrast, a direct spectral partition of the original graph is far from the correct assignment!

Partition into Multiple Clusters

Iterative Bipartition:

Partition the graph into two subgraphs. Then partition each subgraph. Iterates this procedure.

K-Means:

If the number of partitions is fixed and known beforehand, apply the standard K-means (Arthur and Vassilvitskii [2006]) to $\{s(i), i \in [n]\}$. Different clusters represent nodes with different levels of disagreements.

(For more details, see Gao CDC' 21.)

Outline

Which centrality is relevant?

Why degree centrality weights work for these two examples?

Different centralities may be relevant for different applications.

Examples:
page-rank centrality, eigen-centrality, Shapley values, betweenness, etc.

General Centrality-Weighted Opinion Dynamics

Basic Modelling Assumptions

Basic assumptions for general centrality-weighted opinion dynamics:

(i) **Social Conformity:**
Individual on a social network communicate and change their own opinions in the direction to conform with those of their neighbours;

(ii) **Centrality-Weighted Influence:**
Each individual weights these influences from the neighbours proporional to the centrality vector ρ (or time-varying $\rho(t)$).

Degree-Weighted Opinion Dynamics

Centrality-weighted opinion evolution for $x = [x_1, \ldots, x_n]^\top$:

$$\tau\dot{x} = -\bar{L}(t)x, \quad x(0) = x_0 \tag{5}$$

Laplacian matrix $\bar{L}(t)$:

$$\bar{L}(t) = I_n - \bar{A}(t), \quad \text{with } \bar{A}(t) = [\text{diag}(A\rho(t))]^{-1}A\text{diag}(\rho(t)).$$

- $\rho_i(t) > 0$: the centrality of node i on the network
- a_{ij} represents the connection between node j and node i
- $\tau > 0$ is an appropriate time constant

The centrality $\rho(\cdot)$ should be chosen according to the underlying application problems.

Spectral Properties of $\bar{L}(t)$ and $\bar{A}(t)$

Assume the underlying graph $\mathcal{G}(A)$ with the adjacency matrix A is connected and undirected.

Properties

(P1): All the eigenvalues of $\bar{A}(t)$ and $\bar{L}(t)$ are real.

(P2): $\bar{A}(t)$ and $\bar{L}(t)$ are diagonalizable.

(P3): $\bar{L}(t)$ contains only one zero eigenvalue and all the other eigenvalues of $\bar{L}(t)$ are strictly positive.

Hence Partition Alg. (Social Choice Alg.) still works here!

(The partition is possibly time-varying.)

Other Related Aspects

DeGroot formulation with centrality-weighted influence

p_{ki}: the probability of individual i support a given opinion at time k
$p_k = [p_{k1}, \ldots, p_{kn}]$: probability (row) vector

Probability transition: $$p_{k+1} = p_k \bar{A}^{\intercal}(t), \tag{6}$$

where $$\bar{A}_{ij}(t) = \frac{\rho_j(t)}{\sum_{j \in N_i} a_{ij} \rho_j(t)} a_{ij}, \quad i, j \in \{1, ..., n\}.$$

Measure for opinion diversity

- Opinion diversity energy
- Inverse entropy diversity
- Inverse Simpson index

(For more details see Gao CDC'21)

Outline

Conclusion and Future Directions

Conclusion

- Centrality-weighted opinion dynamics
- Network Partition Procedure

Future Directions

- More real-world examples with different types of centralities
- Systematic procedures to learn suitable centralities
- (Equilibrium) state-dependent centralities
- Centralities that allow updates over time

Thank you! Questions?

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