Fixed-Point Centrality for Networks

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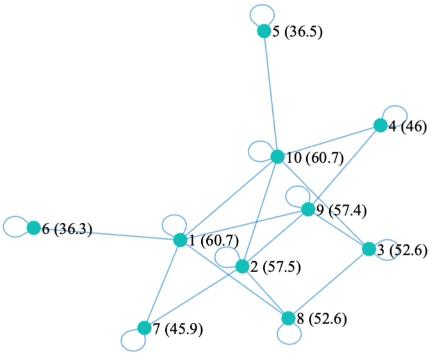
Outline

Introduction: Notion of Centrality

Centrality ($\rho : V \to \mathbb{R}_{\geq 0}$) quantifies the "importance" or "influence" of nodes on networks.



Node	Centrality
1	60.7
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6	36.3

Introduction: Notion of Centrality

Centrality ($\rho : V \to \mathbb{R}_{\geq 0}$) quantifies the "importance" or "influence" of nodes on networks.

Examples:

- Social influence on social networks reflected by eigenvector centrality [Bonacich, 1972]
- Quality of websites modelled by PageRank centrality [Brin and Page, 1998]
- Equilibrium actions in static LQ network games proportional to Katz-Bonacich centrality [Ballester et al., 2006]
- ...

Applications: social, technological and biological networks.

Motivation

- Non-Transferability: Different centralities are defined for different problems
- Centrality Variations: Networks are growing and varying in terms of nodes and (or) connections and hence centrality values may vary accordingly
- New Centrality Notions: Dynamic games on networks/graphons (Gao et al. [2022]; Caines and Huang [2021]) and centrality-weighted opinion dynamics [Gao, 2021]

Questions to Answer:
(a) unify the representations of different centralities?
(b) characterize centrality changes?
(c) identify centralities for dynamic games and opinion models on networks?

Outline

Centrality for Finite Network: Examples

- 1 Eigen Centrality: Assume the largest eigenvalue λ_1 of A is simple

$$\rho_i = \frac{1}{\lambda_1}\sum_{j=1}^{n} a_{ji}\rho_j, \quad i \in [n], \quad \text{i.e.} \quad \rho = \frac{1}{\lambda_1}A^{\mathsf{T}}\rho$$

- 2 Katz-Bonacich Centrality: Let $\alpha \in (0,\ \|A\|_2^{-1})$.

$$\rho_i = \alpha\sum_{j=1}^{n} a_{ji}\rho_j + 1,\ i \in [n], \quad \text{i.e.} \quad \rho = \alpha A^{\mathsf{T}}\rho + \mathbf{1}_n$$

- 3 PageRank: Let $\alpha \in (0, 1)$.

$$\rho_i = \alpha\sum_{j=1}^{n} a_{ji}\frac{\rho_j}{d_j} + \frac{1-\alpha}{n}, \quad i \in [n], \quad \text{i.e.} \quad \rho = \alpha A^{\mathsf{T}}D^{-1}\rho + \frac{1-\alpha}{n}\mathbf{1}_n$$

with $d_j = \sum_{i=1}^{n} a_{ji}$ and $D \triangleq \text{diag}(d_1, ..., d_n)$.

Definition (Permutation Equivariance)

- A mapping $f(\cdot,\cdot) : \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^n$ is permutation equivariant with respect to a permutation map $\pi : [n] \to [n]$ if

$$P_\pi f(A,\rho) = f(P_\pi A P_\pi^\intercal, P_\pi \rho), \quad \forall \rho \in \mathbb{R}^n,\ \forall A \in \mathbb{R}^{n\times n},$$

  where P_π is the permutation matrix corresponding to π.

- A mapping $f(\cdot,\cdot) : \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^n$ is permutation equivariant if it is permutation equivariant w.r.t. all permutation map $\pi : [n] \to [n]$.

Permutation Invariance: $f(A,\rho) = f(P_\pi A P_\pi^\intercal, P_\pi \rho)$

Fixed-Point Centrality for Finite Networks

Definition (Fixed-Point Centrality)

A centrality $\rho : [n] \to \mathbb{R}_{\geq 0}$ is a *fixed-point centrality* for $\mathcal{G}(A)$ associated with the feature space (S^n, d) if there exist

- (a) a permutation equivariant mapping $f(\cdot, \cdot) : \mathbb{R}^{n\times n} \times S^n \to S^n$,
- (b) a permutation equivariant mapping $g(\cdot) : S^n \to \mathbb{R}^n_{\geq 0}$,
- (c) and a unique $x \in S^n$ under the metric d

such that

$$\begin{aligned} x &= f(A, x), \quad x \in S^n, \\ \rho &= g(x), \quad\quad \rho \in \mathbb{R}^n_{\geq 0}. \end{aligned} \tag{1}$$

Note: choices of f and g depend on application contexts.

Results on Fixed-Point Centrality

Proposition

Eigenvector, Katz-Bonacich and PageRank centralities are fixed-point centralities.

For eigenvector centrality,

$$f(A, x) = \frac{1}{\lambda_1} Ax, \quad \text{the fixed-point feature } x \text{ is unique up to its linear span.}$$

For Katz-Bonacich centrality,

$$f(A, x) = \alpha A^{\mathsf{T}} x + \mathbf{1}_n, \quad \alpha \in (0,\ \|A\|_2^{-1}), \quad \text{contraction under 2-norm.}$$

For PageRank centrality, $\alpha \in (0, 1)$,

$$f(A, x) = \alpha A^{\mathsf{T}} \text{diag}(A^{\mathsf{T}} \mathbf{1}_n)^{-1} x + \frac{1-\alpha}{n} \mathbf{1}_n, \quad \text{contraction under 1-norm.}$$

Results on Fixed-Point Centrality

LQG Network Mean Field Game Problem (see [Gao et al., 2022])

Dynamics: $dx_i(t) = (Ax_i(t) + Bu_i(t) + Dz_i(t))dt + \Sigma dw_i(t),$

Cost: $J_i(u_i, u_{-i}) \triangleq \mathbb{E}\int_0^T \left(\|x_i(t) - z_i(t)\|_Q^2 + \|u_i(t)\|_R^2\right)dt$

Network Mean Field : $z_i(t) = \frac{1}{N}\sum_{\ell=1}^{N} m_{q\ell}\frac{1}{|\mathcal{C}_\ell|}\sum_{j\in\mathcal{C}_\ell} x_j(t), \quad i \in \mathcal{C}_q$

Proposition

The equilibrium cost of LQG Network Mean Field Games with a homogenous initial condition, under technical conditions, is a fixed-point centrality.

$$\rho_i = J(z_i), \quad z = \Phi(A, z), \quad z \in (C([0,T]; \mathbb{R}^q))^n.$$

Results on Fixed-Point Centrality

An automorphism of a (directed or undirected) graph $\mathcal{G}(V, E)$ is a permutation map $\pi : V \to V$ that satisfies

$$(i, j) \in E \quad \text{if and only if} \quad (\pi(i), \pi(j)) \in E, \ \forall i, j \in V.$$

Proposition

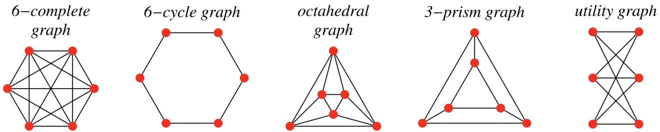
Any fixed-point centrality is permutation invariant with respect to any automorphisms.

Results on Fixed-Point Centrality

A vertex transitive graph is a graph $\mathcal{G}$ satisfying that for any given node pair (i, j), there exists some automorphism map $\phi^{i,j} : [n] \to [n] \in$ such that $\phi^{i,j}(i) = j$.

Proposition (Vertex Transitive Graphs)

All nodes of a vertex transitive graph share the same fixed-point centrality value.



Implication in network games: without computation, one can conclude that equilibrium costs are homogenous among nodes for vertex-transitive graphs.

Results on Fixed-Point Centrality

Centrality Variations

Fixed-Point centralities for two graphs $\mathcal{G}(A)$ and $\mathcal{G}(B)$:

$$\begin{aligned} \rho_A &= g(x_A), \quad x_A = f(A, x_A), \\ \rho_B &= g(x_B), \quad x_B = f(B, x_B). \end{aligned} \tag{2}$$

Theorem

Under Assumption (A1) for the fixed-point centrality, the following holds

$$\|\rho_A - \rho_B\| \leq \frac{L_1 L_g}{1 - L_0(A)} \|A - B\|_{\mathsf{op}} \tag{3}$$

where $\|A\|_{\mathsf{op}} := \sup_{v \neq 0} \frac{\|Av\|}{\|v\|}$.

Implications: convergence of graphs implies convergence of centralities.

Technical Assumption (A1)

(a) There exists $L_1 > 0$ such that for all $x \in \mathcal{U}_f$ (the set of feasible fixed-point features),

$$\|f(A,x) - f(B,x)\| \leq L_1 \|A - B\|_{\text{op}}, \quad \text{with } \|A\|_{\text{op}} := \sup_{v \neq 0} \frac{\|Av\|}{\|v\|} \tag{4}$$

(b) For any matrix A and for any $x \in \mathcal{U}_f$, there exists $L_0(A,x) \geq 0$ such that

$$\|f(A,x_A) - f(A,x)\| \leq L_0(A,x) \|x_A - x\| \quad \text{where } x_A = f(A,x_A); \tag{5}$$

(c) For the given matrix A,

$$L_0(A) := \sup_{x \in \mathcal{U}_f} L_0(A,x) < 1;$$

(d) There exists $L_g > 0$ such that for all $x, v \in \mathcal{U}_f$,

$$\|g(x) - g(v)\| \leq L_g \|x - v\|.$$

Results on Fixed-Point Centrality

Centrality Variations: Centralities as Probability Mass Functions

Proposition

Consider two symmetric matrices A and B. Assume (A1) *and* (A2) *for the fixed-point centrality* (2) *hold. If $|a_{ij}| \leq 1$ and $|b_{ij}| \leq 1$ for all $i, j \in [n]$, then*

$$W_2(\rho_A, \rho_B) \leq \frac{L_1 L_g}{1 - L_0(A)} \sqrt{8\delta_\square(A, B)} \tag{6}$$

where the cut metric is given by

$$\delta_\square(A, B) := \inf_{\pi \in \Pi} \|A^\pi - B\|_\square, \qquad \|A\|_\square := \max_{S \times T \subset [n] \times [n]} \Big| \sum_{i \in S, j \in T} a_{ij} \Big|$$

and Π denotes the set of all permutations from $[n]$ to $[n]$.

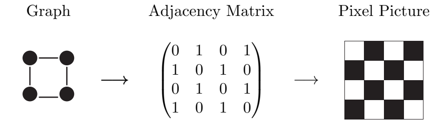
Wasserstein distance: $$W_2(\rho_A, \rho_B) := \left(\inf_{\gamma \in \Gamma(\rho_A, \rho_B)} \int_{\mathcal{X} \times \mathcal{X}} \|x - y\|_2^2 d\gamma(x, y) \right)^{\frac{1}{p}}$$

where $\Gamma(\rho_A, \rho_B)$ denotes the set of joint probability measures with marginals ρ_A and ρ_B.

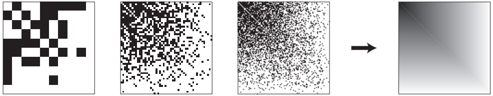
Outline

Centrality for Graphons

Graphons: bounded symmetric measurable function $W : [0,1]^2 \to [0,1]$



$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$



Concept of Graphon [Lovász, 2012]: large/uncertain/growing graphs/graph limits

Eigen, Katz, and PageRank centralities for graphons [Avella-Medina et al., 2018].

Centrality for Graphons: PageRank Example

The *graphon PageRank centrality* is defined as follows:

$$\rho = \alpha \mathbf{A} \odot \mathbf{D}^{-1} \rho + (1 - \alpha)\mathbf{1}, \quad \mathbf{A} \in \mathcal{W}_0, \quad \text{with } \alpha \in (0,1) \tag{7}$$

where $\mathbf{D}(x) = \int_{[0,1]} \mathbf{A}(y,x)dy$, and $(\mathbf{A} \odot \mathbf{D}^{-1})(x) = \frac{\mathbf{A}(x,y)}{\mathbf{D}(y)}$ if $\mathbf{D}(y) \neq 0$, and zero otherwise.

Proposition

The graphon PageRank centrality ρ is a probability density function over $[0,1]$.

Graphon Fixed-Point Centrality: Definition

Definition (Graphon Fixed-Point Centrality)

A centrality $\rho : [0,1] \to \mathbb{R}_{\geq 0}$ is a *fixed-point centrality* for a graphon $\mathbf{A} \in \mathcal{W}_c$ associated with the feature space $(S^{[0,1]}, d)$ if there exist

- a permutation equivariant fixed-point mapping $f(\cdot,\cdot) : \mathcal{W}_c \times S^{[0,1]} \to S^{[0,1]}$,
- a permutation equivariant mapping $g(\cdot) : S^{[0,1]} \to \mathbb{R}_{\geq 0}$,
- a unique function $\mathbf{x} \in S^{[0,1]}$ under the metric d,

such that

$$\begin{aligned} &\mathbf{x} = f(\mathbf{A}, \mathbf{x}), \\ &\rho = g(\mathbf{x}), \quad \rho_\gamma \geq 0, \ \ \gamma \in [0,1]. \end{aligned} \tag{8}$$

Results on Graphon Fixed-Point Centrality

Consider two graphons $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{W}_c$ and

$$\begin{aligned} \mathbf{x_A} &= f(\mathbf{A}, \mathbf{x_A}), \quad \rho_A = g(\mathbf{x_A}), \\ \mathbf{x_B} &= f(\mathbf{B}, \mathbf{x_B}), \quad \rho_B = g(\mathbf{x_B}), \end{aligned} \tag{9}$$

where the feature space $S^{[0,1]}$ is specialized to $L^p([0,1])$ with $p \geq 1$, and the operators $f(\cdot,\cdot)$ and $g(\cdot)$ are specialized to $f(\cdot,\cdot) : \mathcal{W}_c \times L^p([0,1]) \to L^p([0,1])$ and $g(\cdot) : L^p([0,1]) \to L^p([0,1])$.

Theorem

Under Assumption (A3) *for the graphon fixed-point centrality, the following holds*

$$\|\rho_{\mathbf{A}} - \rho_{\mathbf{B}}\| \leq \frac{L_1 L_g}{1 - L_0(\mathbf{A})} \|\mathbf{A} - \mathbf{B}\|_{\text{op}}. \tag{10}$$

Implications: convergence of graphons implies convergence of centralities.

Results on Graphon Fixed-Point Centrality

Centrality Variations: Centralities as Probability Density Functions

Proposition

Consider two graphons $\mathbf{A}$ *and* $\mathbf{B}$ *in* $\mathcal{W}_1$. *Assume* (A3) *and* (A4) *for the graphon fixed-point centrality* (9) *hold. Then the following holds*

$$W_2(\rho_{\mathbf{A}}, \rho_{\mathbf{B}}) \leq \frac{L_1 L_g}{1 - L_0(\mathbf{A})} \sqrt{8\delta_{\square}(\mathbf{A}, \mathbf{B})}. \tag{11}$$

where the cut metric is given by

$$\delta_{\square}(\mathbf{A}, \mathbf{B}) \triangleq \inf_{\phi \in \Phi} \|\mathbf{A}^{\phi} - \mathbf{B}\|_{\square}, \quad \|\mathbf{A}\|_{\square} \triangleq \sup_{S,T \subset [0,1]} \left| \int_{S \times T} \mathbf{A}(x,y) dx dy \right|$$

and Φ *denotes the set of all measure preserving bijections* $\phi : [0,1] \to [0,1]$.

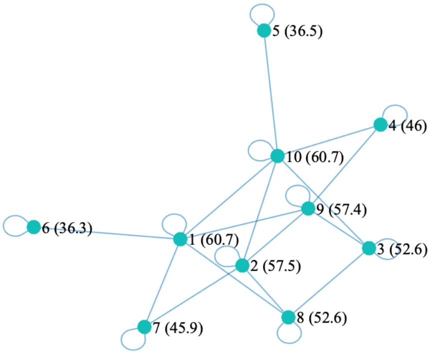
Conclusion

- Fixed-point centralities (for finite and infinite networks)
- Changes of fixed-point centralities with respect to graph changes (a) as vectors (b) as probability distributions.

Future Work

- Exploring connections with games on networks, graph neural networks, MDP, etc.
- Sparse graph limit models (e.g. L^p graphons, graphings).
- Improving upper bounds for centrality variations.
- Ranking variations of fixed-point centralities with respect to network modifications.

Thank You!



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Example: Fixed-point centrality for dynamic games on networks with controlled SIR dynamics.

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Connection with Graph Neural Networks

Fixed-point characterization of Graph Neural Networks
(Gori et al. [2005]; Scarselli et al. [2009])

$$
\begin{aligned}
&\text{feature :} \quad && x = F_\theta(A, x, l), \quad x \in R^{n \times d_x}, \\
&\text{output :} \quad && o = G_\theta(A, x, l_n), \quad o \in \mathbb{R}^{n \times d_o}. \\
&\text{GNN :} \quad && o = \text{GNN}_\theta(A) \\
&\text{error :} \quad && e_w = \frac{1}{m}\sum_{i=1}^{m}(y^{(i)} - \text{GNN}_\theta(A^{(i)}))^2 \\
&\text{data points (input, output):} \quad && (A^{(i)}, y^{(i)}), \quad i \in \{1, ..., m\}
\end{aligned}
\tag{12}
$$

Results on Fixed-Point Centrality

Centrality Variations: Centralities as Probability Mass Functions

Normalization Assumption (A2): $\sum_{i=1}^{n} \rho_i = 1$

Proposition

Under Assumptions (A1) *and* (A2)*, the following holds for the fixed-point centrality in* (2)*:*

$$W_p(\rho_A, \rho_B) \leq \frac{L_1 L_g}{1 - L_0(A)} \inf_{\pi \in \Pi} \|A^\pi - B\|_{\text{op,p}}, \quad \textit{with } \|A\|_{\text{op, p}} := \|A\|_p \tag{13}$$

where

Wasserstein distance: $$W_p(\rho_A, \rho_B) := \left(\inf_{\gamma \in \Gamma(\rho_A, \rho_B)} \int_{\mathcal{X} \times \mathcal{X}} d(x,y)^p d\gamma(x,y) \right)^{\frac{1}{p}}$$

$\Gamma(\rho_A, \rho_B)$*: the set of joint probability measures with marginals* ρ_A *and* ρ_B*.*

Results on Graphon Fixed-Point Centrality

Centrality Variations: Centralities as Probability Density Functions

Proposition

Under Assumptions (A3) *and* (A4)*, the following holds for the fixed-point centrality in* (9)*:*

$$W_p(\rho_{\mathbf{A}}, \rho_{\mathbf{B}}) \leq \frac{L_1 L_g}{1 - L_0(\mathbf{A})} \inf_{\phi \in \Phi} \|\mathbf{A}^\phi - \mathbf{B}\|_{\text{op,p}}, \tag{14}$$

where Φ *denotes the set of all measure preserving bijections from* $[0,1]$ *to* $[0,1]$ *and the operator norm is* $\|\mathbf{A}\|_{\text{op, p}} := \sup_{\mathbf{x}\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$*.*