Optimal Network Location in Infinite Horizon LQG Graphon Mean Field Games

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(1 - 1/3) Finite Games on Networks

We consider games with

- Time is denoted $t \in [0, \infty)$,
- N players, denoted $\mathcal{A}_i, 1 \leq i \leq N < \infty$,
- An n-node network, with adjacency matrix $(g^n_{i,j})_{i,j=1:n}$,
- At node $l \in \{1, ..., n\}$, there is a cluster of players C_l, and

$$N = \sum_{l=1}^{n} |C_l|.$$

(1 - 2/3) Finite Games on Networks

Finite Network Game:

- Players interact via network mean fields, $\{z_t^{i,n}\}$ with $i \in C_h$:

$$z_t^{i,n} = \frac{1}{n}\sum_{l=1}^{n} g_{h,l}^{n}\left(\frac{1}{|C_l|}\sum_{j\in C_l} x_t^j\right), \quad \text{for all } t \geq 0,\ i = 1,\ldots,N.$$

- Each player $\mathcal{A}_i$ chooses a control u^i so as to minimize the cost

$$J^N(u^i, u^{-i}) := \mathbb{E}\int_0^{\infty} e^{-\rho t}\left[r(u_t^i)^2 + \left(x_t^i - z_t^{i,n}\right)^2\right]dt, \tag{1}$$

where $\rho > 0$, u^{-i} denotes the controls of all other players, and

$$dx_t^i = \left(ax_t^i + bu_t^i\right)dt + \sigma dw_t^i, \quad x_0^i \sim \mathcal{N}(m^h, \nu^2), \quad t \geq 0. \tag{2}$$

Note: Uniform network weights $\rightarrow$ standard Mean Field Games.

(1 - 3/3) Finite Games on Networks

Definition (Nash Equilibrium)

A collection of controls, denoted $(u^{i*}, i = 1, ..., N)$, *is a Nash equilibrium if and only if any unilateral deviation from* u^{i*} *to any other control* u^i *yields a higher cost, that is,*

$$J_i^N(u^{i*}, u^{-i*}) \leq J_i^N(u^i, u^{-i*}), \qquad \forall i = 1, ..., N. \tag{3}$$

Remark

- *In general, Nash equilibrium for games on networks gets increasingly harder to obtain as both the players and nodes counts grow.*

(2 - 1/13) Limit Games on Networks

Limit Model

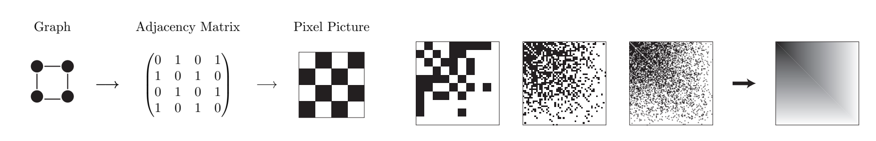
- For non-uniform networks, Graphon Mean Field Games (GMFGs) models the limit games when both,

$$n \to \infty, \quad \text{and} \quad \min_{l \in \{1,\ldots,n\}} |C_l| \to \infty.$$

Assumption (A0)

- *Players in each cluster are exchangeable and their individual impact on the interaction term is negligible.*
- *The networks are modelled by asymptotically dense sequences of graphs* $\{(g^n_{i,j})_{i,j=1:n}\}_{n=1}^{\infty}$ *which converge to a bounded symmetric measurable function (i.e. a graphon).*

(2 - 2/13) Limit Games on Networks



Concept of Graphon for large/uncertain/growing graphs/graph limits (Lovasz'12).

Definition

A graphon is a bounded symmetric measurable function

$$
\begin{aligned}
g : [0,1] \times [0,1] &\to [0,1] \\
: \qquad (\alpha, \beta) &\mapsto g(\alpha, \beta).
\end{aligned}
$$

(2 - 3/13) Limit Games on Networks

Some References on Limit Games on Networks.

- Cluster of players per node : Caines and Huang (2018, 2021), Gao et al (2021) ...
- One player per node : Delarue (2017), Huang et al. (2010), Parise and Ozdaglar (2019), Carmona et al. (2019), Gao et al (2021), Lacker and Soret (2022), Aurell et al. (2022) ...

Motivation for agent cluster per node: exploring agent similarity properties to simplify network analysis (e.g. community detection, neuronal dynamics, epidemics models on networks).

Linear Quadratic Gaussian GMFGs (LQG-GMFGs):

1. Find best responses, $u^{\alpha,o} := (u_t^{\alpha,o})_{t\in[0,T]}$, such that

$$J(u^{\alpha,o}, z^{\alpha}) = \min_{u^{\alpha}\in\mathbb{A}} J(u^{\alpha}, z^{\alpha}) \tag{4}$$
$$= \min_{u^{\alpha}\in\mathbb{A}} \mathbb{E}\int_0^{\infty} e^{-\rho t}\left[r\left(u_t^{\alpha}\right)^2 + \left(x_t^{\alpha} - z_t^{\alpha}\right)^2\right]dt$$

where for all $t \in [0, +\infty),\ \text{and}\ \alpha \in [0,1]$

$$dx_t^{\alpha} = (ax_t^{\alpha} + bu_t^{\alpha})\,dt + \sigma dw_t^{\alpha}, \quad x_0^{\alpha} \sim \mathcal{N}(m^{\alpha}, \nu^2). \tag{5}$$

2. Verify that the optimal states $\{x_t^{\alpha,o},\ t \in [0,\infty),\ \forall\alpha \in [0,1]\}$, satisfy the consistency conditions, $\forall(\alpha, t) \in [0,1] \times [0,\infty)$;

$$z_t^{\alpha} = \int_0^1 g(\alpha,\beta)\mathbb{E}[x_t^{\beta,o}]d\beta. \tag{6}$$

(2 - 5/13) Limit Games on Networks

Proposition (Solvability of LQG-GMFGs)

LQG-GMFGs are solvable whenever there exist solutions, $\{z_t^\alpha, s_t^\alpha, q_t^\alpha, \alpha \in [0,1],\ t \in [0,\infty)\} \subset C_b([0,\infty)) \times L^2([0,1])$ *to*

$$\frac{dz_t^\alpha}{dt} = \left(a - \frac{b^2}{r}\pi\right) z_t^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_t^\beta d\beta, \tag{7}$$

$$\frac{ds_t^\alpha}{dt} = \left(-a + \frac{b^2}{r}\pi + \rho\right) s_t^\alpha + z_t^\alpha, \tag{8}$$

$$z_0^\alpha = \int_0^1 g(\alpha,\beta) m^\beta d\beta,$$

where π *is the positive solution to the Riccati equation*

$$\rho\pi = 2a\pi - \frac{b^2}{r}\pi^2 + 1, \quad r > 0,\ \rho > 0.$$

When the ODEs above has solutions (z, s), it holds that

- the players' best responses are given by,

$$u_t^{\alpha,o} = -\frac{b}{r}\left(\pi x_t^{\alpha,o} + s_t^{\alpha}\right), \quad t \in [0, \infty), \ \alpha \in [0,1],$$

- the players' costs at equilibrium are given by,

$$J(u^{\alpha}, z) = \pi(\nu^2 + (m^{\alpha})^2) + 2 s_0^{\alpha} m^{\alpha} + q_0^{\alpha}, \quad \alpha \in [0,1].$$

where q^{α} satisfies $\frac{dq_t^{\alpha}}{dt} = -\sigma^2 \pi + \frac{b^2}{r}(s_t^{\alpha})^2 - (z_t^{\alpha})^2 + \rho q_t^{\alpha}$

(2 - 7/13) Limit Games on Networks

To solve the ODEs, we derive $(z_\infty^\alpha, s_\infty^\alpha, q_\infty^\alpha)$ from a steady state condition in the infinite horizon,

$$0 = \frac{dz_\infty^\alpha}{dt} = \frac{ds_\infty^\alpha}{dt} = \frac{dq_\infty^\alpha}{dt}, \quad \forall \alpha \in [0,1]. \tag{9}$$

We obtain a family of algebraic equations indexed by $\alpha \in [0,1]$,

$$0 = \left(a - \frac{b^2}{r}\pi\right) z_\infty^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_\infty^\beta d\beta, \tag{10}$$

$$0 = \Big(- a + \frac{b^2}{r}\pi + \rho\Big) s_\infty^\alpha + z_\infty^\alpha, \tag{11}$$

$$0 = -\sigma^2\pi + \frac{b^2}{r}(s_\infty^\alpha)^2 - (z_\infty^\alpha)^2 + \rho q_\infty^\alpha. \tag{12}$$

(2 - 8/13) Limit Games on Networks

Note that the first two algebraic equations are equivalent (almost everywhere on $[0,1]$) to

$$\left[\left(a-\frac{b^2}{r}\pi\right)\left(a-\frac{b^2}{r}\pi\ -\rho\right)I-\frac{b^2}{r}g\right]\circ s_\infty=0, \qquad (13)$$

where

- $(g\circ s_\infty)(\cdot):=\int_0^1 g(\cdot,\beta)s_\infty(\beta)d\beta,$
- I denotes the identity operator from $L^2([0,1])$ to $L^2([0,1])$.

Assumption (A1)

The spectrum of the graphon $g(\cdot,\cdot)$ does not contain the value,

$$\left(\frac{b^2}{r}\right)^{-1}\left(a-\frac{b^2}{r}\pi\right)\left(a-\frac{b^2}{r}\pi\ -\rho\right)$$

Under the above assumption on the eigenvalues of g, the first two algebraic equations admits the (unique) solution in $L^2([0,1])$ with

$$z_\infty^\alpha = 0 = s_\infty^\alpha,\ \alpha \in [0,1]. \tag{14}$$

Then the third algebraic equation admits the solution,

$$q_\infty^\alpha = \frac{\sigma^2 \pi}{\rho},\ \alpha \in [0,1]. \tag{15}$$

Assumption (A2a)

The graphon g is of finite rank, i.e., there exists $L < \infty$ such that

$$g(\alpha, \beta) = \sum_{\ell=1}^{L} \lambda_\ell f_\ell(\alpha) f_\ell(\beta),$$

where f_ℓ is the orthonormal eigenfunction associated with the non-zero eigenvalue λ_ℓ of g.

(2 - 10/13) Limit Games on Networks

Assumption (A2b)

The nonzero eigenvalues of the graphon g satisfy

$$\lambda_\ell < 1 + \frac{a}{b^2} a(a - \rho), \quad \forall \ell \in \{1, \ldots, L\}.$$

Assumption (A2c)

The following inequality holds: $a + \dfrac{1}{\pi} > 0.$

These two assumptions ensure that the equation for (z, s) has a unique solution pair.

(2 - 11/13) Limit Games on Networks

Proposition (Explicit Solutions)

Assume that (A1) and (A2) hold. Then (z, s) *is explicitly given as below* $\forall t \geq 0, \quad \alpha \in [0,1]$,

$$z_t^\alpha = \sum_{l=1}^{L} f_\ell(\alpha) z_t^\ell, \quad s_t^\alpha = -\sum_{l=1}^{L} f_\ell(\alpha) \left(\frac{z_t^\ell}{\theta(\lambda_\ell) + \theta(0)} \right),$$

where for $\ell \in \{1, \ldots, L\}$

$$z_t^\ell = \lambda_\ell \langle m, f_\ell \rangle \exp\left[\left(\frac{\rho}{2} - \theta(\lambda_\ell) \right) t \right] \tag{16}$$

and $\theta(\cdot)$ *is a function defined by*

$$\theta(\tau) := \sqrt{\frac{(\rho - 2a)^2}{4} + (1 - \tau)\frac{b^2}{r}}, \quad \tau \in R. \tag{17}$$

(2 - 12/13) Limit Games on Networks

Sketch of the proof:

- We follow Gao et al. (2021) and define the eigen processes

$$z_t^\ell = \langle z_t, f_\ell \rangle, \quad s_t^\ell = \langle s_t, f_\ell \rangle, \quad t \in [0, \infty],\ \ell \in \{1, ..., L\}.$$

  These processes are solutions to the following ODEs,

$$\frac{dz_t^\ell}{dt} = \left(a - \frac{b^2\pi}{r}\right) z_t^\ell - \lambda_\ell \frac{b^2}{r} s_t^\ell, \quad z_0^\ell = \lambda_\ell \langle m, f_\ell \rangle,\ z_\infty^\ell = 0,$$

$$\frac{ds_t^\ell}{dt} = z_t^\ell + \left(- a + \frac{b^2\pi}{r} + \rho \right) s_t^\ell, \quad s_\infty^\ell = 0.$$

- Differentiating yields the second order ODE for z^ℓ,

$$\frac{d^2 z_t^\ell}{dt} - \rho \frac{dz_t^\ell}{dt} + \left[\lambda_\ell \frac{b^2}{r} - \left(a - \frac{b^2\pi}{r}\right)^2 + \rho \left(a - \frac{b^2\pi}{r}\right)\right] z_t^\ell = 0,$$

  and we explicitly solve for $\{z^\ell,\ \ell = 1, \ldots, L\}$.

- Given the explicit $\{z, s\}$, we derive an explicit $\{q\}$, solving

$$\frac{dq_t^\alpha}{dt} = -\sigma^2\pi + \frac{b^2}{r}(s_t^\alpha)^2 - (z_t^\alpha)^2 + \rho q_t^\alpha, \quad q_\infty^\alpha = \frac{\sigma^2\pi}{\rho}. \quad (18)$$

- Given $\{z, s, q\}$ we compute the players' costs at equilibrium as below,

$$J(u^\alpha, z) = \pi\nu^2 + \pi(m^\alpha)^2) + 2s_0^\alpha m^\alpha + q_0^\alpha, \quad \alpha \in [0, 1].$$

(3 - 1/3) Equilibrium Cost for GMFG

Proposition (Explicit Cost)

Assume (A1)-(A2) hold. Then, the cost at equilibrium is explicitly given below: for almost every $\alpha \in [0,1]$,

$$
\begin{aligned}
J(u^\alpha, z) = \pi\nu^2 + \pi(m^\alpha)^2 + \frac{\sigma^2\pi}{\rho} \\
- \frac{2r}{b^2} m^\alpha \sum_{\ell=1}^{L} f_\ell(\alpha)(\theta(0) - \theta(\lambda_\ell))\langle m, f_\ell\rangle \\
- \sum_{k=1}^{L}\sum_{\ell=1}^{L} f_k(\alpha) f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle \left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)} - 2\right) \\
\frac{1}{\rho}\Big[\lambda_\ell\lambda_k - \frac{r}{b^2}(\theta(0)-\theta(\lambda_\ell))(\theta(0)-\theta(\lambda_k))\Big],
\end{aligned}
$$

where $\theta(\tau) := \sqrt{\frac{(\rho-2a)^2}{4} + (1-\tau)\frac{b^2}{r}}, \ \tau \in R.$

(3 - 2/3) Equilibrium Cost for GMFG

Assumption (A3)

The initial means are non-zero and the same for all nodes, that is,

$$\forall \alpha \in [0,1], \quad m^\alpha = m \neq 0. \tag{19}$$

Proposition (Cost Simplification)

Assume that (A1)-(A3) hold, the equilibrium costs admit the following representation, for every $\alpha \in [0,1]$*,*

$$J(u^\alpha, z) = \pi \left(\nu^2 + m^2 + \frac{\sigma^2}{\rho} \right) - \frac{2b^2}{r} m^2 \int_0^1 \hat{g}(\alpha, \beta) d\beta - m^2 \int_0^1 \tilde{g}(\alpha, \beta \,|\alpha) d\beta.$$

(3 - 3/3) Equilibrium Cost for GMFG

The introduced graphons $\hat{g}(\cdot,\cdot)$ and $\tilde{g}(\cdot,\cdot \mid \alpha)$, $\alpha \in [0,1]$, are finite rank and defined for all $(\epsilon,\beta) \in [0,1]\times[0,1]$ by

$$\hat{g}(\epsilon,\beta) := \sum_{k=1}^{L} \hat{\lambda}_k f_k(\epsilon) f_k(\beta),$$

$$\tilde{g}(\epsilon,\beta \,|\alpha) := \sum_{k=1}^{L} \tilde{\lambda}_k^{\alpha} f_k(\epsilon) f_k(\beta),$$

and for all $k \in \{1,\ldots,L\}$, for all $\alpha \in [0,1]$, the eigenvalues are

$$\hat{\lambda}_k = \theta(0) - \theta(\lambda_k)$$

$$\tilde{\lambda}_k^{\alpha} := \sum_{\ell=1}^{L} f_\ell(\alpha)\langle 1, f_\ell\rangle \left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)} - 2\right)\left(\frac{1}{\rho}\lambda_k\lambda_\ell - \frac{b^2}{\rho r}\hat{\lambda}_k\hat{\lambda}_\ell\right).$$

(4 - 1/2) Minimal Cost Nodes

Proposition (Minimal Cost Nodes)

Assume that (A1)-(A2)-(A3) hold. Any node $\alpha^* \in [0,1]$ *is, almost surely, a node with minimal cost at equilibrium, if and only if,* $\alpha^* \in [0,1]$ *satisfies the condition:*

$$\alpha^* = \underset{\alpha \in [0,1]}{\arg\max} \left[\frac{2r}{b^2} \int_0^1 \hat{g}(\alpha, \beta) d\beta + \int_0^1 \tilde{g}(\alpha, \beta | \alpha) d\beta \right]. \tag{20}$$

(4 - 2/2) Minimal Cost Nodes

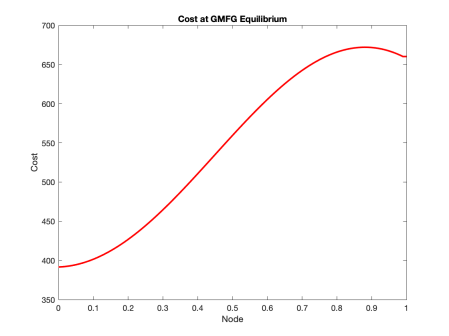
- An example: let the 1-rank approximation of the UA Graphon

$$g(\alpha, \beta) = \lambda f(\alpha) f(\beta), \quad (\alpha, \beta) \in [0, 1]^2. \tag{21}$$

where

$$\lambda = \frac{4}{(3.14)^2}, \qquad f(\alpha) = -\sqrt{2}\cos\left(\frac{\pi}{2}\alpha\right), \quad \alpha \in [0, 1]. \tag{22}$$

- $\alpha^* = 0$ has minimal cost at equilibrium.



(5) Conclusion and Extensions

Conclusion

- We explicitly solved a class of infinite horizon LQG-GMFGs.
- We established the explicit equilibrium cost.
- We found a necessary and sufficient condition for identifying nodes, $\alpha \in [0, 1]$, associated with minimal cost at GMFG equilibrium.

Extensions

- Properties of the newly introduced graphons.
- Links with centrality notions for games on large networks (see Gao CDC'22 Fixed-Point Centrality for Networks).
- Interventions to shape the cost landscape.