

LQG Mean Field Games with Covariance-Matrix-Dependent Cost Coefficients

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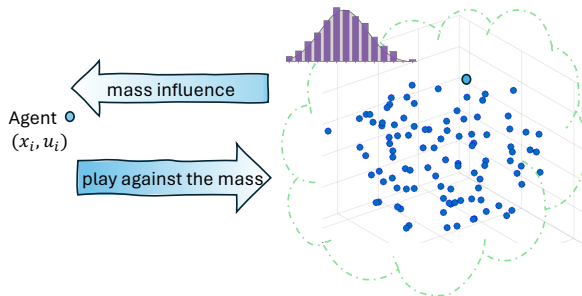
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Mean Field Games: aggregative effect + competition

Mean Field Games (Huang, Malhamé, Caines 2006) (Lasry, Lions, 2007)



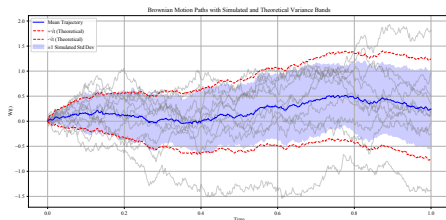
- ▶ Players interact with the empirical state distribution μ^N of N players via dynamics and/or costs
Individual state and control (x_i, u_i) | Empirical distribution $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)}$
- ▶ Objective: overcome dimensionality difficulty + decentralized information

Motivation I: Reducing Variance in Renewable Energy Systems with User Populations

Sources of uncertainties: renewable resources (e.g. solar, wind), user behaviours, etc.

Why reducing the variance?

- load forecasting (to reduce prediction errors)
- demand response (to reduce load variability, particularly during peak-demand periods)



How to design LQ MFG solutions to explicitly take into account of variance, in addition to the mean?

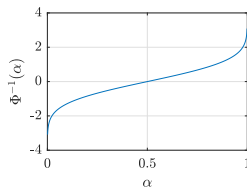
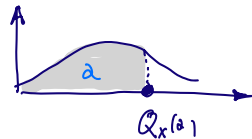
Motivation II: Quantile Mean Field Games Needs Variance

The α -quantile of a random variable X (resp. its distribution F_X) denoted by $\mathcal{Q}_X(\alpha)$ (resp. $\mathcal{Q}_{F_X}(\alpha)$):

$$\mathcal{Q}_X(\alpha) = \mathcal{Q}_{F_X}(\alpha) := \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}, \quad \alpha \in [0, 1], \quad (1)$$

where $F_X(x)$ denotes the cumulative distribution function (CDF) of X .

Gaussian Case: If $X \sim \mathcal{N}(\mu, \sigma^2)$, the quantile function satisfies $\mathcal{Q}_X(\alpha) = \mu + \sigma\Phi^{-1}(\alpha)$ with $\Phi^{-1}(\cdot)$ as the probit function



Related work: Tembine '17, Tchuendom-Malhamé-Caines '19, '24, Gao-Malhamé '25, Tchuendom-Firoozi-Breton '25

Outline

- 1 MFG Strategies with Covariance-Dependent Cost Coefficients
- 2 Existence of MFG Strategies
- 3 Uniqueness of MFG Strategies
- 4 Conclusion

LQG MFG with Covariance-Dependent Cost Coefficients

► Dynamics:

$$dx_i(t) = (Ax_i(t) + Bu_i(t))dt + \Sigma dw_i(t), \quad x_i(t) \in \mathbb{R}^n, u_i(t) \in \mathbb{R}^m, w_i(t) \in \mathbb{R}^q \quad (2)$$

with the initial condition $x_i(0) \sim \mathcal{N}(\mu_0, V_0)$

► Finite Population Cost:

$$J_i(u_i, \bar{x}^N) = \mathbb{E} \int_0^T \left(\|x_i(t) - \bar{x}^N(t)\|_{Q_t^{eq}}^2 + \|u_i(t)\|_R^2 \right) dt + \mathbb{E} \|x_i(T) - \bar{x}^N(T)\|_{Q_T}^2, \quad R > 0, \quad (3)$$

where $\bar{x}^N(t) \triangleq \frac{1}{N} \sum_{i=1}^N x_i(t)$

► Covariance-dependent coefficient:

$$Q_t^{eq} = \Theta(V^N(t)) \geq 0, \quad \forall t \in [0, T], \quad \Theta : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \quad (4)$$

where

$$V^N(t) = \mathbb{E}(x_i(t) - \bar{x}^N(t))(x_i(t) - \bar{x}^N(t))^{\top}.$$

It is assumed that the initial mean μ_0 and initial covariance V_0 are known, and $\|V_0\| > 0$.

LQG MFG with Covariance-Dependent Cost Coefficients

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$$J_i(u_i, \bar{x}) = \mathbb{E} \int_0^T \left(\|x_i(t) - \bar{x}(t)\|_{Q_t^{eq}}^2 + \|u_i(t)\|_R^2 \right) dt + \mathbb{E} \|x_i(T) - \bar{x}(T)\|_{Q_T}^2, \quad R > 0, \quad (3)$$

where $\bar{x}(t) \triangleq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N x_i(t)$

► Covariance-dependent coefficient:

$$Q_t^{eq} = \Theta(V(t)) \geq 0, \quad \forall t \in [0, T], \quad \Theta : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \quad (4)$$

where

$$V(t) = \mathbb{E}(x_i(t) - \bar{x}(t))(x_i(t) - \bar{x}(t))^{\top}.$$

LQG MFG with Covariance-Dependent Cost Coefficients

Best Response

Given the (Gaussian) mean field distribution $(\bar{x}(t), V(t))_{t \in [0, T]}$, the best response for agent i :

$$u_i^*(t) = -R^{-1}B^\top(\Pi_t x_i(t) + s_t) \quad (5)$$

where

$$\begin{aligned} -\dot{\Pi}_t &= A^\top \Pi_t + \Pi_t A - \Pi_t B R^{-1} B^\top \Pi_t + Q_t^{eq}, \quad \Pi_T = Q_T, \\ -\dot{s}_t &= (A - B R^{-1} B^\top \Pi_t)^\top s_t - Q_t^{eq} \bar{x}(t), \quad s_T = -Q_T \bar{x}(T). \end{aligned}$$

LQG MFG with Covariance-Dependent Cost Coefficients

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Under the best response (5), the closed-loop dynamics for a generic agent i are then given by

$$dx_i = (A - B R^{-1} B^\top \Pi_t) x_i dt - B R^{-1} B^\top s_t dt + \Sigma dw_i.$$

Taking the expectation yields the evolution of the mean:

$$d\bar{x} = ((A - B R^{-1} B^\top \Pi_t) \bar{x} - B R^{-1} B^\top s_t) dt, \quad \bar{x}(0) = \mu_0.$$

Note: $\bar{x}_i = \bar{x}$ for all i , since agents are assumed here to have homogeneous parameters.

LQG MFG with Covariance-Dependent Cost Coefficients

Covariance Evolution

Let $e_i = x_i - \bar{x}$. Then

$$de_i = (A - BR^{-1}B^\top \Pi_t)e_i dt + \Sigma dw_i.$$

Applying Itô's lemma for $e_i e_i^\top$ and taking the expectation $V(t) = \mathbb{E}[e_i e_i^\top]$ yield the Lyapunov equation

$$\dot{V}(t) = (A - BR^{-1}B^\top \Pi_t)V(t) + V(t)(A - BR^{-1}B^\top \Pi_t)^\top + \Sigma \Sigma^\top$$

with the initial condition $V(0) = V_0$.

LQG MFG with Covariance-Dependent Cost Coefficients

MFG Solution

Lemma (MFG Solution)

The solution to the limit MFG problem specified by (2), (3) and (4), if exists, satisfies

$$u_i^*(t) = -R^{-1}B^\top(\Pi_t x_i(t) + s_t), \quad t \in [0, T] \quad (6)$$

where

$$\begin{aligned} -\dot{\Pi}_t &= A^\top \Pi_t + \Pi_t A - \Pi_t B R^{-1} B^\top \Pi_t + Q_t^{eq}, \quad \Pi_T = Q_T \\ -\dot{s}_t &= (A - B R^{-1} B^\top \Pi_t)^\top s_t - Q_t^{eq} \bar{x}(t), \quad s_T = -Q_T \bar{x}(T) \\ \dot{\bar{x}}(t) &= (A - B R^{-1} B^\top \Pi_t) \bar{x}(t) - B R^{-1} B^\top s_t, \quad \bar{x}(0) = \mu_0 \end{aligned}$$

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with the constraint

$$Q_t^{eq} = \Theta(V(t)), \quad \forall t \in [0, T]. \quad (7)$$

Simplification using invariance embedding: let $s_t = P_t \bar{x}(t)$.

LQG MFG with Covariance-Dependent Cost Coefficients

MFG Solution via decoupling Riccati equation

Lemma (MFG Solution via Riccati Equation)

The solution to the limit MFG problem specified by (2), (3) and (4), if exists, satisfies

$$u_i^*(t) = -R^{-1}B^\top(\Pi_t x_i(t) + P_t \bar{x}(t)) \quad (8)$$

where $\bar{x}$ satisfies

$$\dot{\bar{x}}(t) = (A - BR^{-1}B^\top(\Pi_t + P_t))\bar{x}(t), \quad \bar{x}(0) = \mu_0,$$

and $\Pi_{(\cdot)}$, $P_{(\cdot)}$ and $V(\cdot)$ satisfy

$$-\dot{\Pi}_t = A^\top \Pi_t + \Pi_t A - \Pi_t B R^{-1} B^\top \Pi_t + Q_t^{eq}, \quad \Pi_T = Q_T \quad (9)$$

$$-\dot{P}_t = (A - BR^{-1}B^\top \Pi_t)^\top P_t + P_t (A - BR^{-1}B^\top \Pi_t) - P_t B R^{-1} B^\top P_t - Q_t^{eq}, \quad P_T = -Q_T, \quad (10)$$

$$\dot{V}(t) = (A - BR^{-1}B^\top \Pi_t)V(t) + V(t)(A - BR^{-1}B^\top \Pi_t)^\top + \Sigma \Sigma^\top, V(0) = V_0 \quad (11)$$

with the constraint $Q_t^{eq} = \Theta(V(t)) \geq 0$.

LQG MFG with Covariance-Dependent Cost Coefficients

Simplifications

Let $H_t := \Pi_t + P_t$. Summing the differential equations for Π and P yields

$$-\dot{H}_t = A^\top H_t + H_t A - H_t B R^{-1} B^\top H_t, \quad H_T = 0. \quad (12)$$

Clearly $H_t = 0$ for all $t \in [0, T]$.

Hence the MFG best response solution:

$$\begin{aligned} u_i^*(t) &= -R^{-1} B^\top (\Pi_t x_i(t) + (H_t - \Pi_t) \bar{x}(t)) \\ &= -R^{-1} B^\top \Pi_t (x_i(t) - \bar{x}(t)). \end{aligned} \quad (13)$$

Remark:

- ▶ Alternative method (e.g. state extension, Huang'10)
- ▶ H_t might not be zero depending on the problem formulation (e.g. $\|x_i - \Gamma \bar{x}\|_{Q_t^{eq}}^2$ with $\Gamma \neq I$ in the cost): see Zhu and Gao MTNS '26.

LQG MFG with Covariance-Dependent Cost Coefficients

Simplifications

Lemma (Simplified MFG Solutions)

The solution to the limit MFG problem specified by (2), (3) and (4), if exists, satisfies

$$u_i^*(t) = -R^{-1}B^\top \Pi_t(x_i(t) - \bar{x}(t)) \quad (14)$$

where

$$-\dot{\Pi}_t = A^\top \Pi_t + \Pi_t A - \Pi_t B R^{-1} B^\top \Pi_t + \Theta(V(t)), \quad \Pi_T = Q_T \quad (15)$$

$$\dot{V}(t) = (A - B R^{-1} B^\top \Pi_t)V(t) + V(t)(A - B R^{-1} B^\top \Pi_t)^\top + \Sigma \Sigma^\top, \quad V(0) = V_0 \quad (16)$$

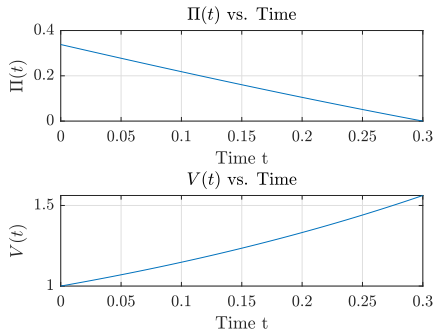
and $\bar{x}$ satisfies

$$\dot{\bar{x}}(t) = A\bar{x}(t), \quad \bar{x}(0) = \mu_0.$$

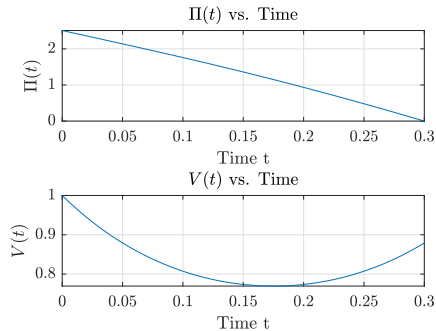
Remark:

Related to covariance steering for linear stochastic systems (Chen-Georgiou-Pavon TAC'15).

Numerical Solutions



(a) $Q^{eq} = Q$



(b) $Q^{eq} = Q(1 + e^{\sqrt{V(t)}\Phi^{-1}(\alpha)})$

Parameters $A = 0.5; B = 1; R = 1; \Sigma^2 = 1; V_0 = 1; Q = 1; \alpha = 0.99; T = 0.3; \mu_0 = 1; Q_T = 0$

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Assumptions for Existence of MFG Solutions

Assumption (A1)

$\Theta(E) \geq 0$, for all $E \in \mathbb{R}^{n \times n}$, and $\Theta(\cdot)$ is continuous with respect to its argument.

Assumption (A2)

$\Theta(\cdot)$ satisfies that for all $E \in \mathbb{R}^{n \times n}$ with $0 < b_1 \leq \|E\| \leq b_2 < \infty$, the following holds

$$\|\Theta(E)\| \leq c_1 + c_2(b_1, b_2)\|E\|,$$

for some $c_1 \in \mathbb{R}$, and $0 \leq c_2(b_1, b_2) < \infty$ (which could depend on b_1 and b_2).

Examples: $\exp(E)$, E^2 and E^{-1} .

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Examples: $\exp(E)$, E^2 and E^{-1} .

Denote $\|A\| = \alpha$, $\|B\| = \beta$, $\|R\| = r > 0$. Let T and M satisfy the inequality

$$T \left[\left(2\alpha M + \frac{\beta^2}{r} M^2 + c_1 \right) + c_2(b_1, b_2) e^{2(\alpha + \frac{\beta^2}{r} M)T} (\|V_0\| + T\|\Sigma\Sigma^\top\|) \right] \leq M. \quad (17)$$

Existence of MFG Solutions

Consider $[0, T]$ and $K := \{\Pi \in C([0, T], \mathbb{R}^n) : \|\Pi\|_\infty \leq M\}$.

Theorem (Existence)

Assume (A1) and (A2) hold. If T and M satisfy the relation (17), then the solution to the limit MFG problem specified by (2), (3) and (4) exists and is given by

$$u_i^*(t) = -R^{-1}B^\top (\Pi_t(x_i(t) - \bar{x}(t))) \quad (18)$$

where

$$\dot{V}(t) = (A - BR^{-1}B^\top \Pi_t)V(t) + V(t)(A - BR^{-1}B^\top \Pi_t)^\top + \Sigma\Sigma^\top, \quad V(0) = V_0 \quad (19)$$

$$-\dot{\Pi}_t = A^\top \Pi_t + \Pi_t A - \Pi_t BR^{-1}B^\top \Pi_t + \Theta(V(t)), \quad \Pi_T = Q_T \quad (20)$$

$$\dot{\bar{x}} = A\bar{x}, \quad \bar{x}(0) = \mu_0. \quad (21)$$

Note: $\Pi \in K$ and $V \in C([0, T], \mathbb{R}^n)$

Proof Idea

Given $\Pi_{(\cdot)} \in C([0, T], \mathbb{R}^n)$, the solution to covariance evolution (16) satisfies

$$V(t) = \Phi(t, 0)V_0\Phi^\top(t, 0) + \int_0^t \Phi(t, s)\Sigma\Sigma^\top\Phi^\top(t, s) ds \quad (22)$$

where $\Phi(t, s)$ is the state transition matrix that satisfies

$$\frac{\partial}{\partial t}\Phi(t, s) = (A - BR^{-1}B^\top\Pi_t)\Phi(t, s), \quad \Phi(s, s) = I, \forall t, s \in [0, T]. \quad (23)$$

Substituting $V(t)$ in Riccati equation (15) yields

$$\begin{aligned} -\dot{\Pi}_t &= A^\top\Pi_t + \Pi_tA - \Pi_tBR^{-1}B^\top\Pi_t \\ &\quad + \Theta\left(\Phi(t, 0)V_0\Phi^\top(t, 0) + \int_0^t \Phi(t, s)\Sigma\Sigma^\top\Phi^\top(t, s) ds\right). \end{aligned}$$

The equivalent integral representation is

$$\begin{aligned} \Pi_t &= \Pi_T + \int_t^T \left[A^\top\Pi_q + \Pi_qA - \Pi_qBR^{-1}B^\top\Pi_q \right. \\ &\quad \left. + \Theta\left(\Phi(q, 0)V_0\Phi^\top(q, 0) + \int_0^q \Phi(q, s)\Sigma\Sigma^\top\Phi^\top(q, s) ds\right) \right] dq. \end{aligned}$$

Proof Idea

Let $\mathcal{T}$ denote the operator on the Banach space of continuous functions $C([0, T], \mathbb{R}^n)$ with the uniform norm $\|\cdot\|_\infty$, defined as follows: for any $\Pi \in C([0, T], \mathbb{R}^n)$,

$$\begin{aligned} \mathcal{T}(\Pi)(t) = & \int_t^T \left[A^\top \Pi_q + \Pi_q A - \Pi_q B R^{-1} B^\top \Pi_q \right. \\ & \left. + \Theta \left(\Phi(q, 0) V_0 \Phi^\top(q, 0) + \int_0^q \Phi(q, s) \Sigma \Sigma^\top \Phi^\top(q, s) ds \right) \right] dq \end{aligned} \quad (24)$$

for all $t \in [0, T]$.

Given $M > 0$, consider the set of uniformly bounded continuous function over $[0, T]$ defined by $K := \{\Pi \in C([0, T], \mathbb{R}^n) : \|\Pi\|_\infty \leq M\}$.

1. K is a closed, bounded, and convex subset of the normed space $C([0, T]; \mathbb{R}^n)$ endowed with the uniform norm
2. $\mathcal{T} : K \rightarrow C([0, T]; \mathbb{R}^n)$ is a compact map such that $\{\mathcal{T}(\Pi) : \Pi \in K\} \subseteq K$ (under the condition (17))

The Schauder fixed-point theorem [Conway1990, p. 150] implies the desired result.

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Assumptions for Uniqueness

Assumption (A3)

The function $\Theta : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ satisfies that for all $E_i \in \mathbb{R}^{n \times n}$ with $0 < b_1 \leq \|E_i\| \leq b_2 < \infty$, $i = 1, 2$, the following hold

$$\|\Theta(E_1) - \Theta(E_2)\| \leq c_3(b_1, b_2)\|E_1 - E_2\|,$$

for some $0 < c_3(b_1, b_2) < \infty$ that depend on b_1 and b_2 .

Let T and M satisfy the following inequality

$$T(2a + 2\frac{\beta^2}{r}M + c_3(b_1, b_2)2e^{3(\alpha + \frac{\beta^2}{r}M)T}\frac{\beta^2}{r}T(\|V_0\| + T\|\Sigma\Sigma^\top\|)) < 1 \quad (25)$$

where $b_1 = e^{-2(\alpha + \frac{\beta^2}{r}M)T}\|V_0\|$, and $b_2 = e^{2(\alpha + \frac{\beta^2}{r}M)T}(\|V_0\| + T\|\Sigma\Sigma^\top\|)$.

LQ MFG with Covariance-Dependent Cost Coefficients

Consider $[0, T]$ and $K := \{\Pi \in C([0, T], \mathbb{R}^n) : \|\Pi\|_\infty \leq M\}$.

Theorem (Uniqueness)

Assume (A1) and (A3) hold. If T and M satisfy the relation (25), then the solution to the limit MFG problem specified by (2), (3) and (4) exists and is unique; furthermore, the MFG solution is given by

$$u_i^*(t) = -R^{-1}B^\top (\Pi_t(x_i(t) - \bar{x}(t))) \quad (26)$$

where $V(t)$, Π_t and $\bar{x}(t)$ satisfies the same equations in the previous theorem for all $t \in [0, T]$.

Proof idea: Under condition (25), we obtain $\|\mathcal{T}(\Pi) - \mathcal{T}(\tilde{\Pi})\|_\infty < \|\Pi - \tilde{\Pi}\|_\infty$. Then apply Banach fixed-point theorem in K .

Note: $\Pi \in K$ and $V \in C([0, T], \mathbb{R}^n)$.

Conclusion

- ▶ LQG MFG strategies with covariance-dependent cost coefficients
- ▶ Sufficient conditions for existence and uniqueness

Extensions and Future work

- ▶ Other types of covariance dependence
e.g. price-coordinated MFG (Dandachly-Gao-Malhamé CDC'26)
- ▶ more explicit conditions for existence and uniqueness of solutions
- ▶ ε -Nash properties

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[Foguen Tchuendom et al.2019; Foguen Tchuendom et al.2024; Tchuendom et al.2025; Gao and Malhamé2025; Zhu and Gao2026; Chen et al.2015]