Networked Control of Coupled Subsystems

Spectral Decomposition and Low-dimensional Solutions

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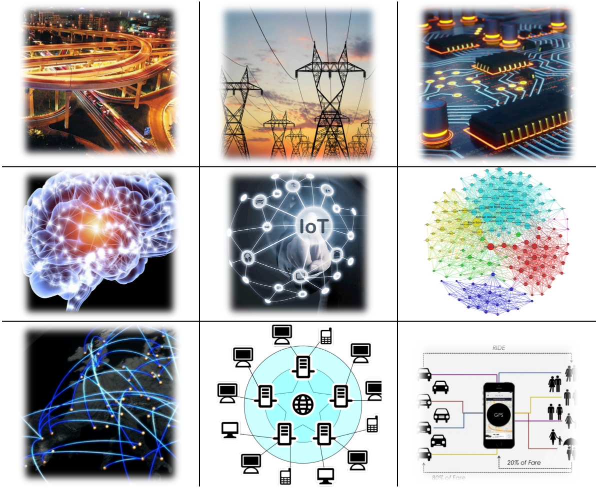
Motivation

Networks are ubiquitous, growing in size and complexity



Online Social Networks, Grid Networks, Transportation Networks, IoT ...

Complex networks:

1. large/growing sizes
2. complex connections
3. dynamics

Salient Features:

1. local states
2. interactions
3. cost

Literature Overview

Low-complexity solutions that scale? [Ho and Mitter, 1976; Sandell et al., 1978]

Various aspects of low-complexity solutions:

- Simplified control objectives (e.g., consensus or synchronization)
  [Olfati-Saber and Murray, 2003; Movric and Lewis, 2013; Arenas et al., 2008]
- Simplified control inputs (e.g., pinning control or ensemble control) [Grigoriev et al., 1997; Wang and Chen, 2002] [Li, 2011]
- Simplified couplings between subsystems (e.g., symmetric interconnections, exchangeable subsystems, or patterned systems)
  [Lunze, 1986; Grizzle and Marcus, 1985; Yang and Zhang, 1995, 1996; Sundareshan and Elbanna, 1991],
  [Madjidian and Mirkin, 2014; Arabneydi and Mahajan, 2017], [Hamilton and Broucke, 2012]
- Approximate optimality as the number of subsystems become large (e.g., mean-field games or graphon-based control)
  [Huang et al., 2003, 2006; Lasry and Lions, 2006; Li and Zhang, 2008] [Gao and Caines, 2017, 2018a,b].

Outline

1. Network of Linear Dynamical Systems
2. Decomposition Method
3. Main Result
4. Illustrative Example
5. Conclusion

Network of Linear Dynamical Systems

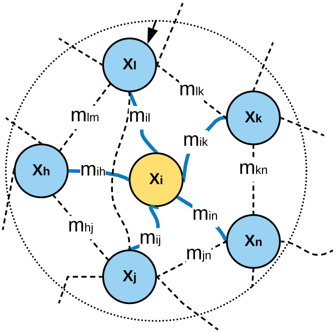
Dynamics

Undirected weighted network: $\mathcal{G}(\mathcal{N}, \mathcal{E}, M)$

Adjacency matrix: $M = [m_{ij}] \in \mathbb{R}^{n \times n}$

Node i: state $x_i(t) \in \mathbb{R}^{d_x}$

control $u_i(t) \in \mathbb{R}^{d_u}$



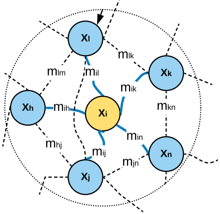
Network influence perceived at node i:

$$z_i(t) = \sum_{j \in \mathcal{N}_i} m_{ij} x_j(t)$$

$$v_i(t) = \sum_{j \in \mathcal{N}_i} m_{ij} u_j(t).$$

Network of Linear Dynamical Systems

Dynamics



The dynamics of node i:

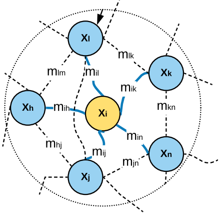
$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + Dz_i(t) + Ev_i(t), \tag{1}$$

Network influence perceived at node i:

$$z_i(t) = \sum_{j \in \mathcal{N}_i} m_{ij} x_j(t) \quad \text{and} \quad v_i(t) = \sum_{j \in \mathcal{N}_i} m_{ij} u_j(t). \tag{2}$$

Network of Linear Dynamical Systems

Control cost



The instantaneous cost at $t \in [0, T)$:

$$c(x(t), u(t)) = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \left[g_{ij} x_i(t)^\intercal Q x_j(t) + h_{ij} u_i(t)^\intercal R u_j(t) \right] \quad (3)$$

Terminal cost:

$$c_T(x(T)) = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} g_{ij} x_i(T)^\intercal Q_T x_j(T), \quad (4)$$

(A1) $Q \geqslant 0$, $Q_T \geqslant 0$ and $R > 0$.

Notation: $G = [g_{ij}]$ and $H = [h_{ij}]$.

Network of Linear Dynamical Systems

Example on cost coupling

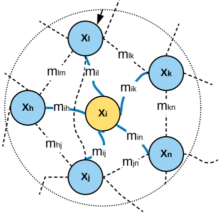
$$G = q_0 I + q_1 M + q_2 M^2 \text{ and } H = r_0 I + r_1 M + r_2 M^2$$

(a) A graph $\mathcal{G}$ (b) 2-hop neighborhood of $\mathcal{G}$

$$M = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M^2 = \begin{bmatrix} 5 & 0 & 5 & 0 \\ 0 & 8 & 0 & 4 \\ 5 & 0 & 5 & 0 \\ 0 & 4 & 0 & 2 \end{bmatrix}.$$

Network of Linear Dynamical Systems

Assumptions on control cost



Cost weight matrices G and H are polynomials of M

$$G = \sum_{k=0}^{K_G} q_k M^k \quad \text{and} \quad H = \sum_{k=0}^{K_H} r_k M^k$$

(A2) For $\ell \in \{0, 1, \dots, L\}$, $q^\ell \geqslant 0$ and $r^\ell > 0$.

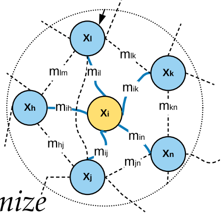
$$q^\ell = \sum_{k=0}^{K_G} q_k (\lambda^\ell)^k \quad \text{and} \quad r^\ell = \sum_{k=0}^{K_H} r_k (\lambda^\ell)^k.$$

(A2) ensures that, $G \geqslant 0$ and $H > 0$.

L: rank of M ; $\lambda^1, \dots, \lambda^L$: non-zero (real) eigenvalues of M.

Network of Linear Dynamical Systems

Control objective



Problem (1)

Choose a control trajectory $u \colon [0, T) \to \mathbb{R}^{d_u \times n}$ *to minimize*

$$J(u) = \int_0^T c(x(t), u(t))dt + c_T(x(T)) \tag{5}$$

subject to the dynamics

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + Dz_i(t) + Ev_i(t), \tag{6}$$

$$z_i(t) = \sum_{j \in \mathcal{N}_i} m_{ij} x_j(t) \quad \text{and} \quad v_i(t) = \sum_{j \in \mathcal{N}_i} m_{ij} u_j(t). \tag{7}$$

Network of Linear Dynamical Systems

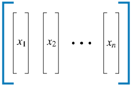
Dynamics: Compact form

"Vectorized" representations of system dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) + Dz(t) + Ev(t). \tag{8}$$

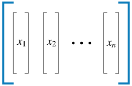
Matrix state and control actions:

$$x(t) = \text{cols}(x_1(t), \dots, x_n(t)) \in \mathbb{R}^{d_x \times n}$$
$$u(t) = \text{cols}(u_1(t), \dots, u_n(t)) \in \mathbb{R}^{d_u \times n}$$



Matrix network influence:

$$z(t) = \text{cols}(z_1(t), \dots, z_n(t)) \in \mathbb{R}^{d_x \times n}$$
$$v(t) = \text{cols}(v_1(t), \dots, v_n(t)) \in \mathbb{R}^{d_u \times n}$$



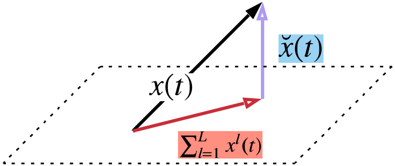
Note:

$$z(t) = x(t)M^{\intercal} = x(t)M \quad \text{and} \quad v(t) = u(t)M^{\intercal} = u(t)M.$$

Outline

Decomposition

Main idea:



Step 1: Project the state $x(t)$ into L orthogonal eigendirections:

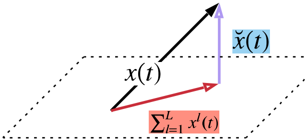
$$\{x^{\ell}(t)\}_{\ell=1}^{L} \quad \text{and} \quad \breve{x}(t) = x(t) - \sum_{\ell=1}^{L} x^{\ell}(t)$$

and similar decompositions for the control inputs.

Step 2: Solve $L+1$ decoupled problems (in dynamics and cost)

Decomposition

Spectral factorization



Spectral Factorization:

$$M = \sum_{\ell=1}^{L} \lambda^\ell w^\ell w^{\ell\intercal}. \qquad (9)$$

For $\ell \in \{1, \ldots, L\}$, define

$$\begin{aligned} \text{Eigenstate}: \quad & x^\ell(t) = x(t) w^\ell w^{\ell\intercal} \\ \text{Eigencontrol}: \quad & u^\ell(t) = u(t) w^\ell w^{\ell\intercal} \end{aligned}$$

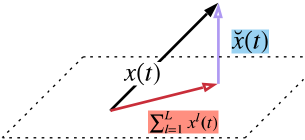
Eigen Dynamics:

$$\dot{x}^\ell(t) = (A + \lambda^\ell D) x^\ell(t) + (B + \lambda^\ell E) u^\ell(t), \qquad (10)$$

Note that $M w^\ell w^{\ell\intercal} = \lambda^\ell w^\ell w^{\ell\intercal}$.

Decomposition

Projections and decoupled dynamics



Auxiliary state and control actions:

$$\breve{x}(t) = x(t) - \sum_{\ell=1}^{L} x^{\ell}(t) \quad \text{and} \quad \breve{u}(t) = u(t) - \sum_{\ell=1}^{L} u^{\ell}(t).$$

Auxiliary Dynamics:

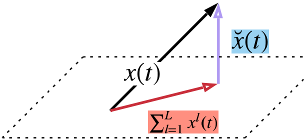
$$\dot{\breve{x}}(t) = A\breve{x}(t) + B\breve{u}(t). \tag{11}$$

Eigen Dynamics:

$$\dot{x}^{\ell}(t) = (A + \lambda^{\ell} D)x^{\ell}(t) + (B + \lambda^{\ell} E)u^{\ell}(t), \tag{12}$$

Decomposition

Local representation of decoupled dynamics



Decoupled "local" auxiliary dynamics:

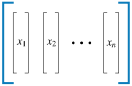
$$\dot{\breve{x}}_i(t) = A\breve{x}_i(t) + B\breve{u}_i(t), \quad i \in \mathcal{N}. \tag{13}$$

Decoupled "local" eigen dynamics:

$$\dot{x}_i^\ell(t) = (A + \lambda^\ell D)x_i^\ell(t) + (B + \lambda^\ell E)u_i^\ell(t), \quad i \in \mathcal{N}. \tag{14}$$

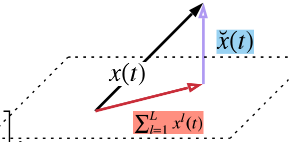
Note:

$$x^\ell(t) = \operatorname{cols}(x_1^\ell(t), \dots, x_n^\ell(t)),$$
$$u^\ell(t) = \operatorname{cols}(u_1^\ell(t), \dots, u_n^\ell(t)).$$



Decomposition

Cost decoupling



$$J(u) = \sum_{i \in \mathcal{N}} \Big[\breve{J}_i(\breve{u}_i) + \sum_{\ell=1}^{L} J_i^\ell(u_i^\ell) \Big].$$

Proposition (Cost Decoupling)

The instantaneous cost may be written as

$$c(x(t), u(t)) = \langle x(t), Qx(t) \rangle_G + \langle u(t), Ru(t) \rangle_H,$$

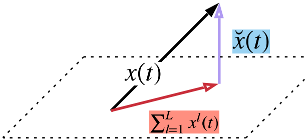
which can be simplified as follows:

$$\langle x(t), Qx(t) \rangle_G = \sum_{i \in \mathcal{N}} \Big[q_0 \breve{x}_i(t)^\intercal Q \breve{x}_i(t) + \sum_{\ell=1}^{L} q^\ell x_i^\ell(t)^\intercal Q x_i^\ell(t) \Big],$$

$$\langle u(t), Ru(t) \rangle_H = \sum_{i \in \mathcal{N}} \Big[r_0 \breve{u}_i(t)^\intercal R \breve{u}_i(t) + \sum_{\ell=1}^{L} r^\ell u_i^\ell(t)^\intercal R u_i^\ell(t) \Big].$$

Decomposition

Decoupled Problem



Auxiliary Problem:

$$\dot{\breve{x}}_i(t) = A\breve{x}_i(t) + B\breve{u}_i(t), i \in \mathcal{N}$$

$$\breve{J}_i(\breve{u}_i) = \int_0^T \left[q_0 \breve{x}_i(t)^\intercal Q \breve{x}_i(t) + r_0 \breve{u}_i(t)^\intercal R \breve{u}_i(t) \right] dt + q_0 \breve{x}_i(T)^\intercal Q \breve{x}_i(T).$$

Eigen Problem:

$$\dot{x}_i^\ell(t) = (A + \lambda^\ell D) x_i^\ell(t) + (B + \lambda^\ell E) u_i^\ell(t), i \in \mathcal{N}$$

$$J_i^\ell(u_i^\ell) = \int_0^T \left[q^\ell x_i^\ell(t)^\intercal Q x_i^\ell(t) + r^\ell u_i^\ell(t)^\intercal R u_i^\ell(t) \right] dt + q^\ell x_i^\ell(T)^\intercal Q x_i^\ell(T).$$

This decomposition for mean-field coupling ($L = 1$) is used in [Arabneydi and Mahajan, 2017].

Outline

Main result

Theorem (Optimal Solution)

For $\ell \in \{1, \dots, L\}$, let $P^\ell \colon [0, T] \to \mathbb{R}^{d_x \times d_x}$ be the solution to the Riccati equation

$$
\begin{aligned}
-\dot{P}^\ell(t) &= (A + \lambda^\ell D)^\intercal P^\ell(t) + P^\ell(t)(A + \lambda^\ell D) \\
&\quad - P^\ell(t)(B + \lambda^\ell E)(r^\ell R)^{-1}(B + \lambda^\ell E)^\intercal P^\ell(t) + q^\ell Q, \quad P^\ell(T) = q^\ell Q_T.
\end{aligned} \tag{15}
$$

Similarly, let $\breve{P} \colon [0, T] \to \mathbb{R}^{d_x \times d_x}$ be the solution to the Riccati equation

$$
-\dot{\breve{P}}(t) = A^\intercal \breve{P}(t) + \breve{P}(t)A - \breve{P}(t)B(r_0 R)^{-1}B^\intercal \breve{P}(t) + q_0 Q, \quad \breve{P}(T) = q_0 Q_T. \tag{16}
$$

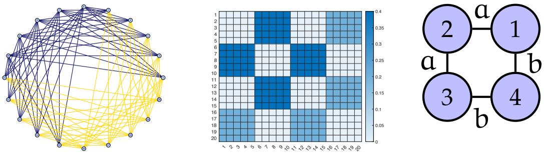
Then, under assumptions (A1) and (A2), the optimal control strategy for Problem 1 is given by

$$
u_i(t) = -\breve{K}(t)\breve{x}_i(t) - \sum_{\ell=1}^{L} K^\ell(t) x_i^\ell(t), \tag{17}
$$

$$
\breve{K}(t) = (r_0 R)^{-1} B^\intercal \breve{P}(t), \quad K^\ell(t) = (r^\ell R)^{-1}(B + \lambda^\ell E)^\intercal P^\ell(t).
$$

Outline

Illustrative example



$$\begin{bmatrix} 0 & a & 0 & b \\ a & 0 & a & 0 \\ 0 & a & 0 & b \\ b & 0 & b & 0 \end{bmatrix}$$

Figure: Graphs with adjacency matrices $M_{20} = M \otimes \frac{1}{5}\mathbb{1}_{5\times 5}$ and M

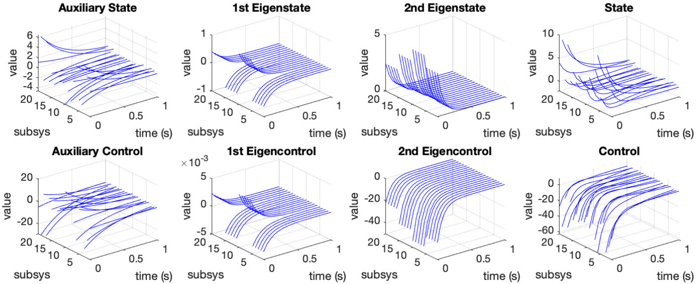
$L = \text{rank}(M) = \text{rank}(M_{20}) = 2.$

Consider the following couplings in the cost

$$G = I - 2M_{20} + M_{20}^2 \quad \text{and} \quad H = I. \tag{18}$$

Illustrative example

Parameters: $a = 2$ and $b = 1$. $d_x = d_u = 1$;
$A = 2, B = 1, D = 3, E = 0.5, Q = 5, Q_T = 6, R = 2.$



The evolutions of the corresponding eigenstates and the auxiliary states along with the eigencontrols and the auxiliary controls

Outline

Conclusion

Low-complexity solution to optimal networked control of coupled subsystems (in dynamics and cost) via an underlying weighted graph.

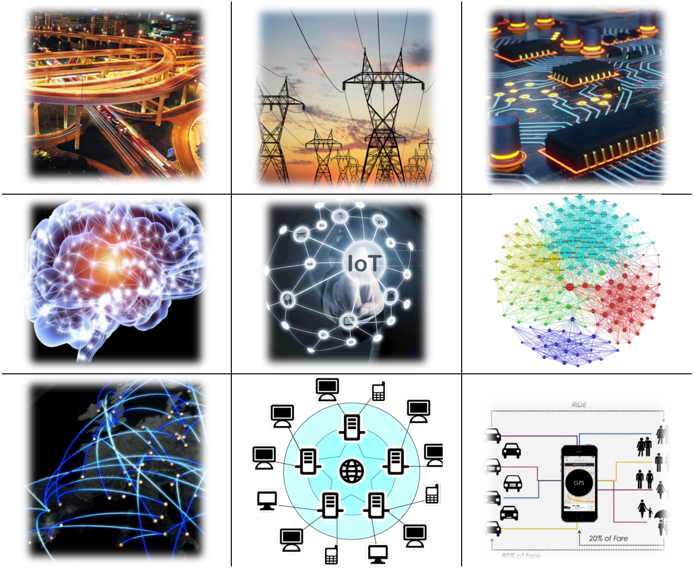
Complexity:

	# of Riccati	Dim of Riccati
Naive soln	1	$nd_x \times nd_x$
Decoupling soln	$L + 1$	$d_x \times d_x$

Note $L \leqslant n$.

Future directions: directed network, heterogeneous local dynamics, uncertainties in dynamics and network structures, approximations ...

Thank you!



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