Optimal and Approximate Solutions to Linear Quadratic Regulation of a Class of Graphon Dynamical Systems

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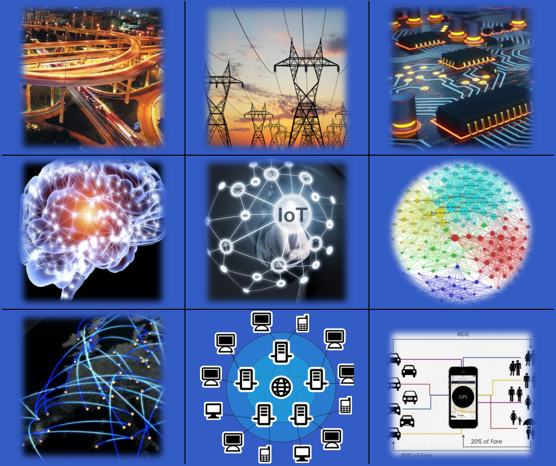
Motivation

Networks are everywhere, growing in size and complexity.



Background

Challenges in controlling complex networks of dynamical systems:

- Large or growing number of nodes
- Complex connections
- Dynamics

Analysis problem

- Controllability, observability and control energy metric, etc.
  (Liu et al., 2011), (Yan et al., 2015),(Pasqualetti et al., 2014)

Synthesis problem

- Simplified control objectives (e.g., consensus or synchronization)
  (Olfati-Saber and Murray, 2003; Movric and Lewis, 2013; Arenas et al., 2008)
- Simplified control inputs (e.g., pinning control or ensemble control)
  (Grigoriev et al., 1997; Wang and Chen, 2002) (Li, 2011)
- Simplified couplings between subsystems (e.g., symmetric interconnections, exchangeable subsystems, or patterned systems)
  (Lunze, 1986; Grizzle and Marcus, 1985; Yang and Zhang, 1995, 1996; Sundareshan and Elbanna, 1991), (Madjidian and Mirkin, 2014; Arabneydi and Mahajan, 2017), (Hamilton and Broucke, 2012)

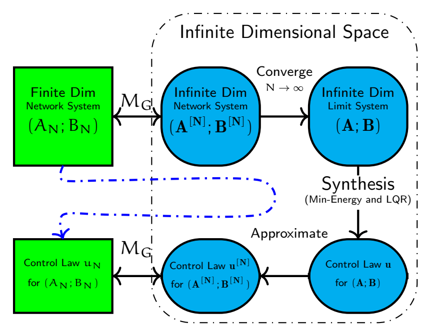
Background

Graphon control: approximate control of dynamical systems on arbitrary-size complex networks

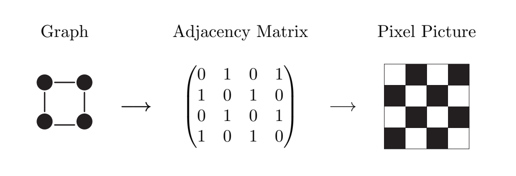
- Mathematical framework for dynamical systems on arbitrary-size networks: Graphon control systems (SG PEC: CDC17, CDC18, MTNS18, CDC19, TAC20)
- Systems and control theory for complex networks of linear dynamical systems:
  (a) Graphon state-to-state control (SG PEC: CDC17, TAC20)
  (b) Centralized graphon linear quadratic regulation (SG PEC: CDC18, MTNS18, TAC20)
  (c) **Collaborative graphon linear quadratic regulation** (SG PEC: CDC19)

Graphon Control Methodology (SG PEC: CDC17,18,19,TAC20)

- Convergence
- Control synthesis
- Approximation



Program

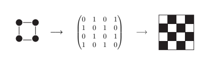


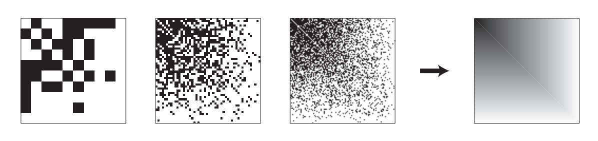
Graph, Adjacency Matrix, Pixel Picture (Lovász, 2012)

The whole pixel picture is presented in a unit square $[0, 1] \times [0, 1]$, so the square elements have sides of length $\frac{1}{N}$, where N is the number of nodes.

Introduction to Graphons

Graph Sequence Converging to Graphon





Graph Sequence Converging to its Limit (Lovász, 2012)

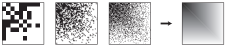
Graphons: bounded symmetric Lebesgue measurable functions

$$\mathbf{W} : [0,1]^2 \to [0,1]$$

interpreted as weighted graphs on the vertex set $[0,1]$.

Notations of Spaces

$$\tilde{\mathbf{G}}_{\mathbf{0}}^{\mathbf{sp}} := \{\mathbf{W} : [0,1]^2 \to [0,1]\}$$
$$\tilde{\mathbf{G}}_{\mathbf{1}}^{\mathbf{sp}} := \{\mathbf{W} : [0,1]^2 \to [-1,1]\}$$
$$\tilde{\mathbf{G}}^{\mathbf{sp}} := \{\mathbf{W} : [0,1]^2 \to \mathbb{R}\}$$



Cut norm: $$\|\mathbf{W}\|_{\Box} := \sup_{M,T \subset [0,1]} \left| \int_{M \times T} \mathbf{W}(x,y)\,dx\,dy \right|$$

Cut metric: $$\delta_{\Box}(\mathbf{W},\mathbf{V}) := \inf_{\phi} \|\mathbf{W}^{\phi} - \mathbf{V}\|_{\Box}, \qquad *^{1}$$

Theorem (Lovász (2012))

The graphon spaces $(\mathbf{G}_{\mathbf{0}}^{\mathbf{sp}}, \delta_{\Box})$ *and any closed bounded subset of* $(\mathbf{G}_{\mathbb{R}}^{\mathbf{sp}}, \text{ffi}_{\Box})$ *are compact.*

By compactness, infinite sequences of graphons will necessarily possess one or more sub-sequential limits under the cut metric.

[1] $\mathbf{W}^{\phi}(x,y) = \mathbf{W}(\phi(x),\phi(y))$

Introduction to Graphons

Graphons as Operators (Lovász, 2012)

Graphon $\mathbf{W} \in \tilde{\mathbf{G}}_1^{\mathbf{sp}}$ as an operator: $\mathbf{W} : L^2[0,1] \to L^2[0,1]$

Operation: $$[\mathbf{W}\mathbf{v}](x) = \int_0^1 \mathbf{W}(x,\alpha)\mathbf{v}(\alpha)d\alpha \qquad \mathbf{v} \in L^2[0,1]$$

Operator Product: $$[\mathbf{U}\mathbf{W}](x,y) = \int_0^1 \mathbf{U}(x,z)\mathbf{W}(z,y)dz, \quad \mathbf{U}, \mathbf{W} \in \tilde{\mathbf{G}}_1^{\mathbf{sp}}$$

Norm relations :

$$\|\mathbf{W}\|_{\mathsf{op}} \leqslant \|\mathbf{W}\|_2, \qquad \|\mathbf{W}\|_\square \leqslant \|\mathbf{W}\|_{\mathsf{op}} \leqslant \sqrt{8\|\mathbf{W}\|_\square}.$$

See (Gao and Caines, 2019), (Janson, 2010; Parise and Ozdaglar, 2018)

Graphon operators are Hilbert-Schmidt operators

(Rudin, 1991; J Mercer, 1909; Szegedy, 2011)

Let $\mathbb{A} = (\alpha_0 \mathbb{I} + \mathbf{A})$ with $\mathbf{A} \in \tilde{\mathbf{G}}_{\mathbf{1}}^{\mathbf{sp}}$. Then

$$[\mathbb{A}\mathbf{v}](\cdot) = \alpha_0 \mathbf{v}(\cdot) + \int_0^1 \mathbf{A}(\cdot, \eta) v(\eta) d\eta, \quad \mathbf{v} \in L^2_{[0,1]}.$$

$\mathbb{A} \in \mathcal{G}^1_{\mathcal{A}\mathcal{I}}$ is a bounded linear operator and hence generates the uniformly continuous semigroup

$$S_{\mathbb{A}}(t) := e^{\mathbb{A}t} = \sum_{k=0}^{\infty} \frac{t^k \mathbb{A}^k}{k!}. \tag{1}$$

The initial value problem of the graphon differential equation

$$\dot{\mathbf{y}}_{\mathbf{t}} = \mathbb{A}\mathbf{y}_{\mathbf{t}}, \quad \mathbf{y}_{\mathbf{0}} \in L^2[0,1], \quad 0 \leqslant t \leqslant T \tag{2}$$

has a solution given by

$$\mathbf{y}_{\mathbf{t}} = e^{\mathbb{A}t}\mathbf{y}_{\mathbf{0}}, \quad \mathbf{y}_{\mathbf{t}} \in L^2[0,1], \quad 0 \leqslant t \leqslant T. \tag{3}$$

Program

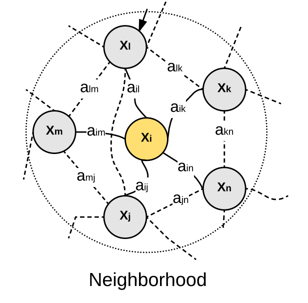
Networks of Linear Systems and Their Limits

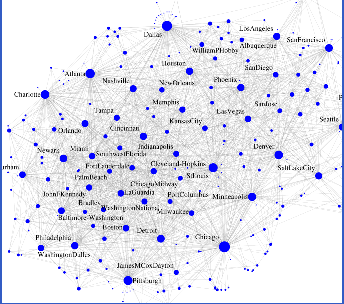
Linear Network System

The dynamics of the i^{th} agent in the network

$$\dot{x}_t^i = \alpha_0 x_t^i + \frac{1}{N}\sum_{j=1}^{N} a_{ij} x_t^j + \beta_0 u_t^i + \frac{1}{N}\sum_{j=1}^{N} b_{ij} u_t^j, \tag{4}$$

$$t \in [0, T], \quad \alpha_0, \beta_0 \in \mathbb{R}, \quad x_t^i, u_t^i \in \mathbb{R},$$





Networks of Linear Systems and Their Limits

Linear Network Systems Described by Graphons

Dynamics:

$$\dot{\mathbf{x}}_t^{[N]} = (\alpha_0 \mathbb{I} + \mathbf{A}^{[N]})\mathbf{x}_t^{[N]} + (\beta_0 \mathbb{I} + \mathbf{B}^{[N]})\mathbf{u}_t^{[N]}, \quad t \in [0, T],$$
$$\alpha_0, \beta_0 \in \mathbb{R}, \quad \mathbf{x}_t^{[N]}, \mathbf{u}_t^{[N]} \in L^2_{pwc\,[0,1]}, \quad \mathbf{A}^{[N]}, \mathbf{B}^{[N]} \in \tilde{\mathbf{G}}_1^{sp} \tag{5}$$

where

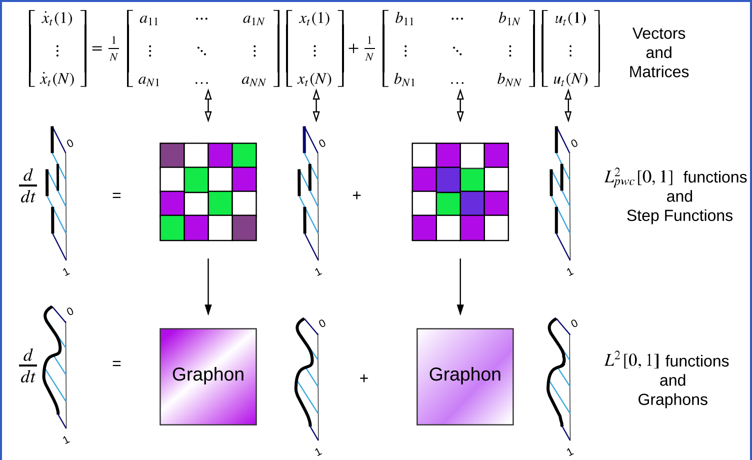
$$\mathbf{A}^{[N]}(\vartheta, \varphi) = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbb{1}_{P_i}(\vartheta)\mathbb{1}_{P_j}(\varphi) a_{ij}, \quad (\vartheta, \varphi) \in [0,1]^2 \tag{6}$$

$$\mathbf{x}_t^{[N]}(\vartheta) = \sum_{i=1}^{N} \mathbb{1}_{P_i}(\vartheta) x_t^i, \quad \forall \vartheta \in [0,1] \tag{7}$$

$\mathbb{1}_{P_i}(\cdot)$: the indicator function; $L^2_{pwc\,[0,1]}$: the set of all piece-wise constant functions in $L^2_{[0,1]}$

Networks of Linear Systems and Their Limits

Linear Network Systems Described by Graphons



Program

Graphon LQR

Graphon System with Non-compact System Operator

We formulate the graphon linear system $(\mathbb{A}; \mathbb{B})$ as follows:

$$\dot{\mathbf{x}}_t = \mathbb{A}\mathbf{x}_t + \mathbb{B}\mathbf{u}_t, \quad t \in [0, T], \tag{8}$$

where $\mathbb{A} = (\alpha_0 \mathbb{I} + \mathbf{A})$ with $\mathbf{A} \in \tilde{\mathbf{G}}_{\mathbf{1}}^{\mathbf{sp}}$ and $\alpha_0 \in \mathbb{R}$, $\mathbb{B} \in \mathcal{L}(L^2[0, 1])$, $\mathbf{x}_t \in L^2[0, 1]$ is the system state at time t, and $\mathbf{u}_t \in L^2[0, 1]$ is the control input at time t.

Proposition (Bensoussan et al. (2007))

The system $(\mathbb{A}; \mathbb{B})$ *in* (8) *has a unique mild solution* $\mathbf{x} \in C([0, T]; L^2[0, 1])$ *for any* $\mathbf{x_0} \in L^2[0, 1]$ *and any* $\mathbf{u} \in L^2([0, T]; L^2[0, 1])$.

Graphon LQR

Control Objective

Objective: $\min_{u} J(\mathbf{u}) = \int_0^T c_t(\mathbf{x}_t, \mathbf{u}_t)dt + c_T(\mathbf{x}_T),$

where $c_t(\mathbf{u}_t, \mathbf{x}_t) = \langle \mathbf{x}_t, \mathbf{Q}\mathbf{x}_t \rangle + \langle \mathbf{u}_t, \mathbf{u}_t \rangle, c_T(\mathbf{x}_T) = \langle \mathbf{x}_T, \mathbf{P}_0\mathbf{x}_T \rangle$

subject to system constrains in (8).

Graphon LQR

Assumptions

1. $\mathbf{Q}$ and $\mathbf{P}_0$ are linear operators on $L^2[0,1]$ that are Hermitian and non-negative (i.e., $\mathbf{Q}, \mathbf{P}_0 \geqslant 0$).
2. The graphon $\mathbf{A}$ as an operator has a finite number (d) eigenfunctions corresponding to the non-zero eigenvalues.

$$\mathbf{A}(x,y) = \sum_{i=1}^{d} \lambda_\ell \mathbf{f}_\ell(x)\mathbf{f}_\ell(y), \quad (x,y) \in [0,1]^2. \tag{9}$$

3. $\mathbb{B}$ is in $\mathcal{PO}(\mathbf{A})$ and it is given by

$$\mathbb{B} = \mathrm{poly}_{\mathbf{B}}(\mathbf{A}) := \sum_{k=0}^{b_L} \beta_k \mathbf{A}^k, \quad b_L \geqslant 0.$$

4. $\mathbf{Q}$ and $\mathbf{P}_0$ are in $\mathcal{PO}(\mathbf{A})$, represented by

$$\mathbf{Q} = \mathrm{poly}_{\mathbf{Q}}(\mathbf{A}) := \textstyle\sum_{k=0}^{h} q_k \mathbf{A}^k, \quad h \geqslant 0,$$

$$\mathbf{P}_0 = \mathrm{poly}_{\mathbf{P}_0}(\mathbf{A}) := \textstyle\sum_{k=0}^{r} z_k \mathbf{A}^k, \quad r \geqslant 0.$$

Graphon LQR

Existence and Uniqueness of Solutions

Riccati equation:

$$\dot{\mathbf{P}} = \mathbb{A}^{\mathsf{T}}\mathbf{P} + \mathbf{P}\mathbb{A} - \mathbf{P}\mathbb{B}\mathbb{B}^{\mathsf{T}}\mathbf{P} + \mathbf{Q}, \quad \mathbf{P}(0) = \mathbf{P}_0 \tag{10}$$

The optimal control $\mathbf{u}^*$:

$$\mathbf{u}_t^* = -\mathbb{B}^{\mathsf{T}}\mathbf{P}(T-t)\mathbf{x}_t^*, \quad t \in [0, T] \tag{11}$$

The closed loop equation:

$$\begin{aligned} &\dot{\mathbf{x}}_t = \mathbb{A}\mathbf{x}_t - \mathbb{B}\mathbb{B}^{\mathsf{T}}\mathbf{P}(T-t)\mathbf{x}_t, \\ &t \in [0, T], \mathbf{x}_0 \in L^2[0, 1]. \end{aligned} \tag{12}$$

Proposition (Bensoussan et al. (2007))

Under **Assumption 1***, there exists a unique solution to the Riccati equation* (10) *and furthermore there exists a unique optimal solution pair* $(\mathbf{u}^*, \mathbf{x}^*)$ *as given in* (11) *and* (12).

Introduce the auxiliary state and control as:

$$\breve{\mathbf{x}}_t = \mathbf{x}_t - \sum_{l=1}^{d} \bar{\mathbf{x}}_t^l, \quad \breve{\mathbf{u}}_t = \mathbf{u}_t - \sum_{l=1}^{d} \bar{\mathbf{u}}_t^l \tag{13}$$

$\bar{\mathbf{x}}_t^l = \langle \mathbf{x}_t, \mathbf{f}_l \rangle \mathbf{f}_l$: projection of $\mathbf{x}_t$ into the eigendirection $\mathbf{f}_l$
$\bar{\mathbf{u}}_t^l = \langle \mathbf{u}_t, \mathbf{f}_l \rangle \mathbf{f}_l$: projection of $\mathbf{u}_t$ into the eigendirection $\mathbf{f}_l$

Graphon LQR

Decoupled LQR Problems

1. Eigensystem LQR problems

$$\begin{cases} \dot{\bar{\mathbf{x}}}^\ell_t = (\alpha_0 + \lambda_\ell)\bar{\mathbf{x}}^\ell_t + \mathrm{poly}_{\mathbf{B}}(\lambda_\ell)\bar{\mathbf{u}}^\ell_t, \\ \bar{J}^\ell(\bar{\mathbf{u}}^\ell) = \int_0^T \bar{c}^\ell_t(\bar{\mathbf{u}}^\ell_t, \bar{\mathbf{x}}^\ell_t)dt + \bar{c}^\ell_T(\bar{\mathbf{x}}^\ell_T), 1 \leqslant l \leqslant d \end{cases} \tag{14}$$

where $\bar{c}^\ell_t(\bar{\mathbf{u}}^\ell_t, \bar{\mathbf{x}}^\ell_t) = \mathrm{poly}_{\mathbf{Q}}(\lambda_\ell)\|\bar{\mathbf{x}}^\ell_t\|_2^2 + \|\bar{\mathbf{u}}^\ell_t\|_2^2$ and $\bar{c}^\ell_T(\bar{\mathbf{x}}^\ell_T) = \mathrm{poly}_{\mathbf{P}_0}(\lambda_\ell)\|\bar{\mathbf{x}}^\ell_T\|_2^2$;

2. Auxiliary system LQR problem

$$\begin{cases} \dot{\breve{\mathbf{x}}}_t = \alpha_0\breve{\mathbf{x}}_t + \beta_0\breve{\mathbf{u}}_t, \\ \breve{J}(\breve{\mathbf{u}}) = \int_0^T \breve{c}_t(\breve{\mathbf{u}}_t, \breve{\mathbf{x}}_t)dt + \breve{c}_T(\breve{\mathbf{x}}_T), \end{cases} \tag{15}$$

where $\breve{c}_t(\breve{\mathbf{u}}_t, \breve{\mathbf{x}}_t) = q_0\|\breve{\mathbf{x}}_t\|_2^2 + \|\breve{\mathbf{u}}_t\|_2^2$ and $\breve{c}_T(\breve{\mathbf{x}}_T) = z_0\|\breve{\mathbf{x}}_T\|_2^2$.

See (Gao, Mahajan CDC19) for direct methods on finite networks. Decoupling method with mean field couplings: (Arabneydi, Mahajan CDC15, ArXiv16)

Graphon LQR

Decoupled LQR Problems

Lemma (SG, PEC, CDC19')

If **Assumptions 1-4** *are satisfied, then solving the optimal control problems* (14) *and* (15) *is equivalent to solving the original optimal control problem defined by* (8) *and* (14). *Moreover, the optimal control solution exists and is unique.*

Theorem (SG, PEC, CDC19')

If **Assumptions 1-4** *are satisfied, then the optimal control law is given by*

$$\mathbf{u}_t = -\beta_0 \mathbf{L}_{T-t} \check{\mathbf{x}}_t - \sum_{\ell=1}^{d} \mathsf{poly}_{\mathbf{B}}(\lambda_\ell) \mathbf{M}^{\ell}_{T-t} \mathbf{x}^{\ell}_t, \tag{16}$$

where $\mathbf{L} := \{\mathbf{L}_t : t \in [0, T]\}$ *is the solution to the Riccati equation*

$$\dot{\mathbf{L}}_t = 2\alpha_0 \mathbf{L}_t - \beta_0^2 \mathbf{L}_t^2 + q_0 \mathbb{I}, \quad \mathbf{L}_0 = z_0 \mathbb{I}, \tag{17}$$

and $\mathbf{M}^\ell := \{\mathbf{M}^\ell_t : t \in [0, T]\}$ *is the solution to the Riccati equation*

$$\begin{aligned} \dot{\mathbf{M}}^\ell_t &= 2(\alpha_0 + \lambda_\ell)\mathbf{M}^\ell_t - \mathsf{poly}_{\mathbf{B}}(\lambda_\ell)^2 (\mathbf{M}^\ell_t)^2 + \mathsf{poly}_{\mathbf{Q}}(\lambda_\ell)\mathbb{I}, \\ \mathbf{M}^\ell_0 &= \mathsf{poly}_{\mathbf{P}_0}(\lambda_\ell)\mathbb{I}, \quad 1 \leqslant l \leqslant d. \end{aligned} \tag{18}$$

Theorem (SG, PEC, CDC19')

If **Assumptions 1-4** *are satisfied, then the localized optimal control law for the* γ^{th} *subsystem with* $\gamma \in [\underline{\gamma}, \overline{\gamma}] \subset [0,1]$ *is given by*

$$\mathbf{u}_t(\gamma) = -\beta_0 L_{T-t} \breve{\mathbf{x}}_t(\gamma) - \sum_{\ell=1}^{d} \mathrm{poly}_{\mathbf{B}}(\lambda_\ell) M^{\ell}_{T-t} \bar{x}^{\ell}_t \mathbf{f}_\ell(\gamma), \tag{19}$$

where $L := \{L_t : t \in [0,T]\}$ *is the solution to the scalar Riccati equation*

$$\dot{L}_t = 2\alpha_0 L_t - \beta_0^2 L_t^2 + q_0, \quad L_0 = z_0, \tag{20}$$

and $M^\ell := \{M^\ell_t : t \in [0,T]\}$ *is the solution to the scalar Riccati equation*

$$\begin{aligned} \dot{M}^\ell_t &= 2(\alpha_0 + \lambda_\ell) M^\ell_t - \mathrm{poly}_{\mathbf{B}}(\lambda_\ell)^2 (M^\ell_t)^2 + \mathrm{poly}_{\mathbf{Q}}(\lambda_\ell), \\ M^\ell_0 &= \mathrm{poly}_{\mathbf{P}_0}(\lambda_\ell), \quad 1 \leqslant l \leqslant d. \end{aligned} \tag{21}$$

Information Required:
For a representative subsystem $\gamma \in [\underline{\gamma}, \overline{\gamma}] \subset [0, 1]$

1. all the eigenvalues of $\mathbf{A}$ and the value of the respective eigenfunctions at its index location, that is, λ_ℓ, $\mathbf{f}_\ell(\gamma)$ for all $1 \leqslant \ell \leqslant d$;
2. the projections of the initial state $\mathbf{x}_0$ onto each eigenfunction direction, that is, $\bar{x}_0^\ell = \langle \mathbf{x}_0, \mathbf{f}_\ell \rangle$ for all $1 \leqslant \ell \leqslant d$;
(Preserving private information)
3. its own state $\mathbf{x}_t(\gamma)$.

Complexity:
Solving $(d + 1)$ scalar Riccati equations VS infinite dimensional Riccati equations

Finite Spectral Approximation

$$\mathbf{A} = \sum_{i=1}^{\infty} \lambda_\ell \mathbf{f}_\ell(x)\mathbf{f}_\ell(y) \approx \mathbf{A}_L := \sum_{\ell=1}^{L} \lambda_\ell \mathbf{f}_\ell(x)\mathbf{f}_\ell(y), \quad (x,y) \in [0,1]^2 \quad (22)$$

Proposition (SG, PEC, CDC19')

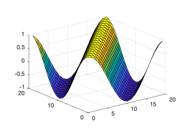
Assume $\mathrm{poly}_{\mathbf{B}}(\mathbf{A}) = \beta_0 \mathbb{I}$*. If the localized optimal control law* (19) *is applied with the approximation of* $\mathbf{A}$ *by* $\mathbf{A}_L$ *given in* (22) *and the observation of eigenstates* $x^\ell, 1 \leqslant \ell \leqslant L$ *by all subsystems is in real time, then*

$$\frac{\widetilde{x}_t^h}{\bar{x}_t^h} = \exp\Big(-\beta_0^2 \int_0^T (\widetilde{M}_t^h - M_t^h) dt \Big), \quad h > L,$$

where

$$\dot{\widetilde{M}}_t^h = 2\alpha_0 \widetilde{M}_t^h - (\beta_0 \widetilde{M}_t^h)^2 + q_0, \quad \widetilde{M}_0^h = z_0$$
$$\dot{M}_t^h = 2(\alpha_0 + \lambda_h) M_t^h - \beta_0^2 (M_t^h)^2 + \mathrm{poly}_{\mathbf{Q}}(\lambda_h), \quad M_0^h = \mathrm{poly}_{\mathbf{P}_0}(\lambda_h).$$

Numerical Example



Parameters:

$\alpha_0 = 2,\ \mathbf{A}(x, y) = \cos(2\pi(x - y)), \quad \forall (x, y) \in [0, 1]^2,\ \mathrm{poly}_{\mathbf{B}}(s) = 1 + \frac{1}{2}s,$
$\mathrm{poly}_{\mathbf{Q}}(s) = (1 - s)^2,\ \mathrm{poly}_{\mathbf{P}_0}(s) = (1 - s)^2,$

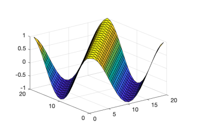
Localized optimal control:

$$\mathbf{u}_t(\gamma) = -L_{T-t}\check{\mathbf{x}}_t(\gamma) - \frac{\sqrt{2}}{4} M^{\ell}_{T-t}(\bar{x}^1_t \sin 2\pi\gamma + \bar{x}^2_t \cos 2\pi\gamma), \gamma \in [\underline{\gamma}, \overline{\gamma}] \subset [0, 1] \tag{23}$$

where $\check{\mathbf{x}}_t(\gamma) = \mathbf{x}_t(\gamma) - \sqrt{2}\bar{x}^1_t \sin 2\pi\gamma - \sqrt{2}\bar{x}^2_t \cos 2\pi\gamma$ and

$$\dot{L}_t = 4L_t - L_t^2 + 1, \quad L_0 = 1,$$
$$\dot{M}^{\ell}_t = 5M^{\ell}_t - \frac{1}{4}(M^{\ell}_t)^2 + \frac{1}{4}, \quad M^{\ell}_0 = 1, \quad \ell \in \{1, 2\}, t \in [0, T].$$

Numerical Example



Eigenfunctions:

$$\mathrm{f}_1 = \sqrt{2}\sin 2\pi(\cdot) \text{ and } \mathrm{f}_2 = \sqrt{2}\cos 2\pi(\cdot); \quad \lambda_1 = \lambda_2 = \frac{1}{2}$$

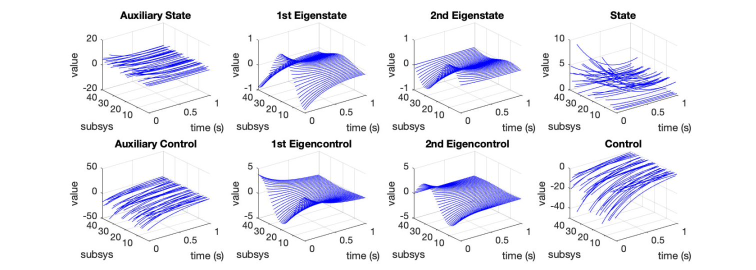


Figure: The simulation demonstration runs on the corresponding step function system based on the uniform partition of size 40. The initial states are generated randomly.

2 scalar Riccati equations VS a 40×40 matrix Riccati equation

Program

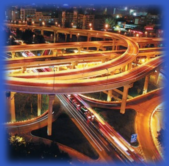
Conclusion

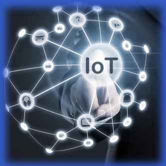
1. **Graphon couplings in cost and dynamics**: we obtain the optimal solution to the linear quadratic regulation of a class of graphon dynamical systems where the dynamics and cost functions share the same underlying graphon structure.
2. **Locally computed solution**: the solution can be computed and implemented locally
3. **Low complexity:** the complexity of the solution depends on the number of nonzero eigenvalues of the underlying graphon.

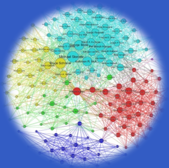
Important aspects includes:

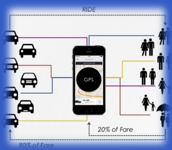
1. Subspace Decomposition
2. Graphon Linear Quadratic Gaussian
3. Graphon Mean Field Games (Caines, Huang CDC18' CDC19')
4. Graphon Control with Non-linear Local Dynamics
5. Control of Time Varying Graphon Dynamical Systems
6. Graphon Control Applications
7. ...

Thank you!









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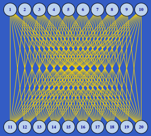
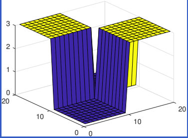
Simulation Example 1

Bipartite Network Example



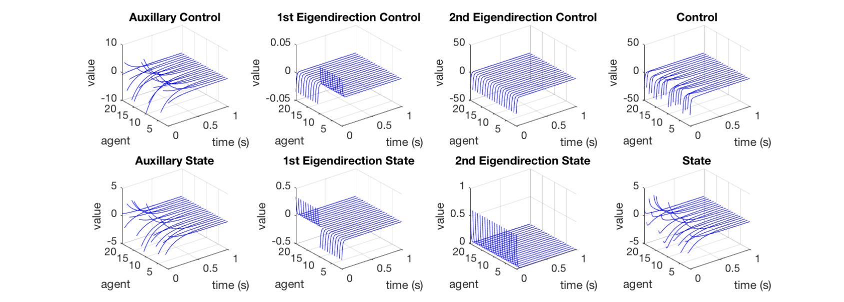


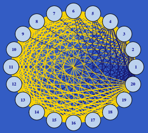
$\mathrm{A}_t = 2; \mathrm{B}_t = 4; \mathrm{D}_t = 3; \mathrm{Q}_t = 7;$

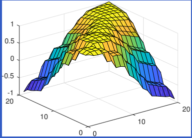
$\mathrm{Q}_T = 5; \mathrm{H}_t = 6; \mathrm{H}_T = 2; \mathrm{R}_t = 3;$

$\mathrm{poly}_r(s) = 1 + 2s + s^2;$

$\mathrm{poly}_q(s) = 1 + 2s + s^2;$







$$A_t = 2; B_t = 4; D_t = 3; Q_t = 7;$$
$$Q_T = 5; H_t = 6; H_T = 2; R_t = 3;$$
$$\mathrm{poly}_r(s) = 1 + 2s + s^2;$$
$$\mathrm{poly}_q(s) = 1 + 2s + s^2;$$